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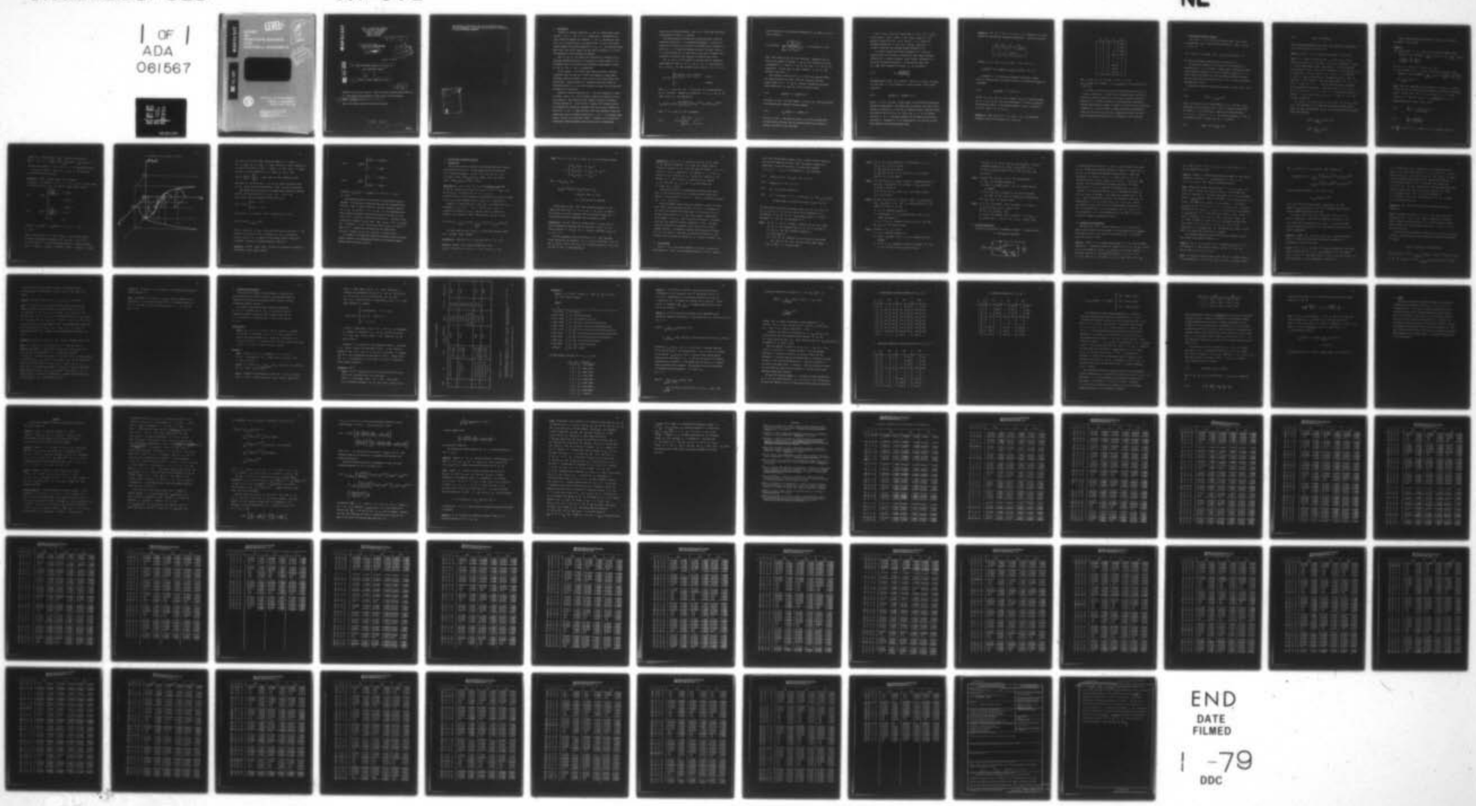

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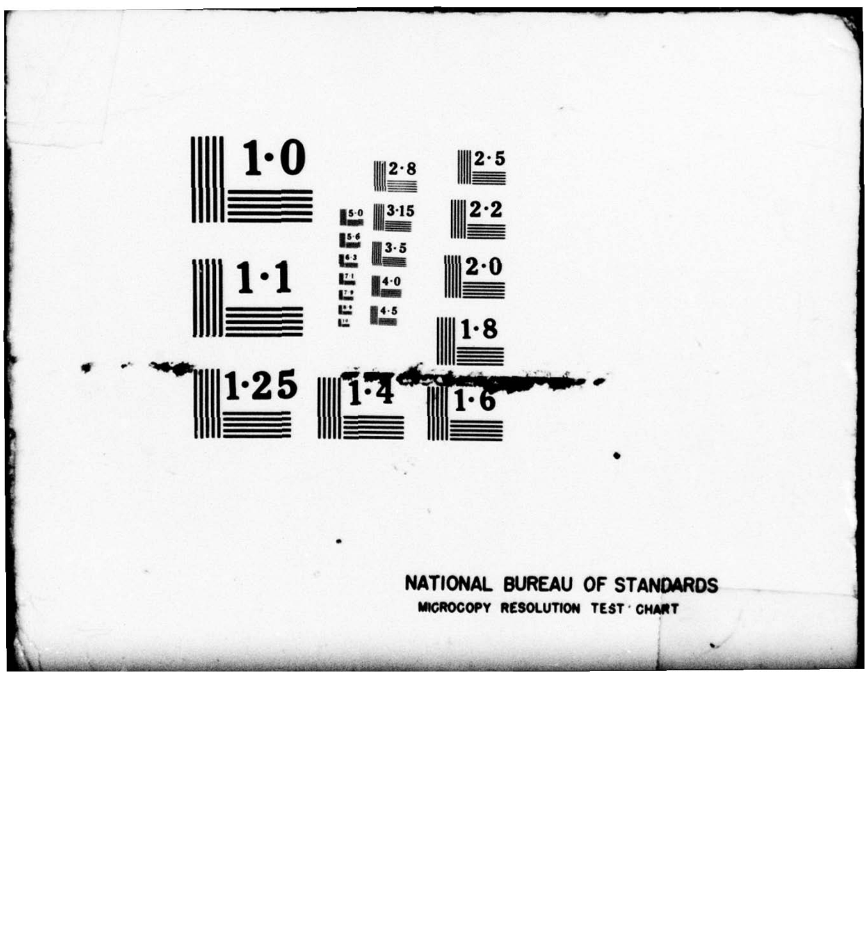
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Psub1 to Psub2

(6) EXACT CONFIDENCE INTERVALS FOR $p_1 - p_2$ IN
2×2 CONTINGENCY TABLES

2 by 2

by

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1. Introduction

Consider two binomial populations π_1 and π_2 having success probabilities $0 < p_1 < 1$ and $0 < p_2 < 1$ respectively. Experimenters have used a variety of measures for comparing π_1 and π_2 including the odds ratio $\psi \equiv p_1(1-p_2)/(1-p_1)p_2$, the relative risk $\rho \equiv p_1/p_2$ and the difference between the success probabilities $\Delta \equiv p_1 - p_2$. See Cornfield (1956), Gart (1971), Gail (1973), Dunnett and Gent (1977), and Katz et. al. (1977) for comparisons of these measures and examples.

When p_1 and p_2 are unknown the statistician is interested in constructing confidence intervals for the measure of interest based on a 2×2 contingency table of data formed from independent random samples of sizes N_1 and N_2 from π_1 and π_2, respectively.

For the measure ψ both exact (small sample) confidence intervals (Cornfield (1956) and Katz et al. (1977)) and asymptotic (large sample) confidence intervals (Cornfield (1956) and Gart (1971) among others) have been devised. Computer programs for implementing the exact methods of Cornfield and Katz et. al. have been published by Thomas (1971) and Baptista and Pike (1977) respectively.

Asymptotic confidence intervals for ρ have been proposed by a number of authors. Katz et. al. (1977) contains a summary and comparison of five such intervals. There have also been a number of asymptotic confidence intervals proposed for Δ (Gart (1971)). Buhrman (1977) proposed methods for designing and analyzing an experiment which yields exact confidence intervals for ρ and Δ. Thomas and Gart (1977) have published a method for constructing "exact" confidence intervals for ρ and Δ in undesigned experiments. However, as we show below, their Δ intervals do not satisfy the conditional confidence guarantee they claim nor even a corresponding

unconditional confidence guarantee. Katz et. al. (1977) give conditional counterexamples for their ρ intervals.

This paper proposes two methods for constructing exact $100(1-\alpha)\%$ confidence intervals for Δ in undesigned experiments. It will be indicated how the methods can be modified to determine exact confidence intervals for ρ. We begin by reviewing that part of the basic theory of 2×2 tables required for our later work; we then give two examples which illustrate the problem of the intervals in Thomas and Gart (1977).

Let X_1 and X_2 be the numbers of successes based on independent random samples of sizes N_1 and N_2 from π_1 and π_2 respectively and let $\underset{\sim}{X} = (X_1, X_2)$. The joint probability mass function of X_1 and X_2 is

$$f(x_1,x_2) = \begin{cases} \binom{N_1}{x_1}\binom{N_2}{x_2} p_1^{x_1} q_1^{N_1-x_1} p_2^{x_2} q_2^{N_2-x_2}, & (x_1,x_2) \in S \\ 0, & \text{otherwise} \end{cases}$$

where $q_i \equiv 1 - p_i (i=1,2)$ and $S \equiv \{(y_1,y_2) | y_i$ is an integer between 0 and N_i inclusive$\}$. Regarded as a function of $\Delta = p_1 - p_2$ and p_1 for fixed $(x_1,x_2) \in S$, the likelihood is

$$P_{p_1,\Delta}[X_1=x_1, X_2=x_2] = \binom{N_1}{x_1}\binom{N_2}{x_2} p_1^{x_1}(1-p_1)^{N_1-x_1}(p_1-\Delta)^{x_2}(1+\Delta-p_1)^{N_2-x_2}$$

where $-1 < \Delta < 1$ and $p_1 \in I(\Delta)$ is given by

$$(1.1) \qquad I(\Delta) \equiv \begin{cases} (0, 1+\Delta), & -1 < \Delta < 0 \\ (0,1), & \Delta = 0 \\ (\Delta,1), & 0 < \Delta < 1. \end{cases}$$

Also of interest is the conditional distribution of X_1 given $X_1 + X_2 = m$ which is given by

$$(1.2)\quad g(j|m,\psi) = \frac{\binom{N_1}{j}\binom{N_2}{m-j}\psi^j}{\sum_{\ell=\ell(m)}^{u(m)}\binom{N_1}{\ell}\binom{N_2}{m-\ell}\psi^\ell}, \quad j = \ell(m), \ell(m)+1, \ldots, u(m)$$

where $\ell(m) \equiv \max\{0, m-N_2\}$ and $u(m) \equiv \min\{m, N_1\}$. Expression (1.2) is valid for any integer m between 0 and $N_1 + N_2$ inclusive; $g(\cdot|m,\psi)$ is degenerate at x_1 equal to 0 and $N_1 + N_2$ when $m = 0$ and $N_1 + N_2$ respectively. In other cases $g(\cdot|m,\psi)$ depends on p_1 and p_2 only through the odds ratio ψ.

Both Cornfield (1956) and Katz et. al. (1977) have proposed methods for constructing exact two-sided $100(1-\alpha)\%$ confidence intervals $(\psi_L(\underset{\sim}{X}), \psi_U(\underset{\sim}{X}))$ for ψ. If $P_\psi[\cdot|m\,]$ denotes a probability calculated under distribution (1.2) then both of their intervals satisfy

$$(1.3)\qquad P_\psi[\psi_L(\underset{\sim}{X}) < \psi < \psi_U(\underset{\sim}{X})|m] \geq 1-\alpha$$

for every $\psi \in (0,\infty)$ and every integer $0 \leq m \leq N_1 + N_2$. Hence they satisfy the (unconditional) confidence interval guarantee

$$(1.4)\qquad P_{p_1,\Delta}[\psi_L(\underset{\sim}{X}) < \psi < \psi_U(\underset{\sim}{X})] \geq 1-\alpha$$

for all p_1 and Δ. The method of Katz et. al. will be reviewed since it yields shorter intervals than the Cornfield method and the tables of Section 2 are based on their intervals.

Fix $m \in \{0,1,\ldots,N_1 + N_2\}$ and for each $\psi \in (0,\infty)$ let $\Lambda = \Lambda_\psi(m)$ be the subset of $\{\ell(m),\ldots,u(m)\}$ so that $P_\psi[X_1 \in \Lambda_\psi|m] \geq 1-\alpha$ and $g(k|m,\psi) \leq g(j|m,\psi)$ for all $k \notin \Lambda_\psi$ and all $j \in \Lambda_\psi$. Define the confidence interval for ψ to be $\{\psi \in (0,\infty)|X_1 \in \Lambda_\psi(X_1+X_2)\}$. It is easy to check that the resulting interval satisfies (1.3).

In a series of papers Gart (1971), McDonald et. al. (1974) and Thomas and Gart (1971) attempt to produce exact small sample confidence intervals for Δ based on intervals $(\psi_L(\underset{\sim}{X}), \psi_U(\underset{\sim}{X}))$ satisfying (1.3) and on the following relationships. To construct the upper limit $\Delta_U = \Delta_U(\underset{\sim}{X})$ first determine the solution x_U of the equation

$$(1.5) \qquad \psi_U = \frac{x_U(x_U+N_2-m)}{(m-x_U)(N_1-x_U)}$$

satisfying $\max\{0, m-N_2\} \leq x_U \leq \min\{m,N_1\}$. Then let $\Delta_U \equiv x_U/N_1 - (m-x_U)/N_2$. The lower limit Δ_L can be obtained in a similar fashion. These authors claim that

$$(1.6) \qquad P_\psi[\Delta_L(\underset{\sim}{X}) < \Delta < \Delta_U(\underset{\sim}{X})|m] \geq 1-\alpha$$

for all $\psi \in (0,\infty)$ and hence $(\Delta_L(\underset{\sim}{X}), \Delta_U(\underset{\sim}{X}))$ also satisfies the probability guarantee unconditionally. This method has two problems. The first is that Equation 1.5 defining x_U is derived on asymptotic grounds (Cornfield (1956)). The second is that if $m \neq N_1$ or $m \neq N_2$ then Δ_L or Δ_U are bounded away from -1 or 1. This second problem will be examined in more detail in Section 4. Consequenty it is not surprising that (1.6) need not hold for small samples as the following example shows.

Example 1.1: Let $N_1 = 2 = N_2$, $m = 1$ and $\alpha = .01$. McDonald et. al. (1974) compute the following 99% confidence intervals for Δ based Gart's method.

x_1	x_2	Δ_L	Δ_U
0	1	-1	.4808
1	0	-.4808	+1

Consider $p_1 = 3/4$ and $p_2 = 1/4$ so that $\Delta = 1/2$ and $\psi = 9$.

$$P_{\psi=9}[\Delta_L(\underset{\sim}{X}) < 1/2 < \Delta_U(\underset{\sim}{X})|1] = P_{\psi=9}[X_1=1,\ X_2=0|1] = .90 < .99.$$

As Example 1.1 is a conditional probability calculation it might still be conjectured that this method satisfies the unconditional probability requirement

(1.7) $$P_{p_1,\Delta}[\Delta_L(\underset{\sim}{X}) < \Delta < \Delta_U(\underset{\sim}{X})] \geq 1-\alpha$$

for all $\Delta \in (-1,1)$ and $p_1 \in I(\Delta)$ provided (Δ_L, Δ_U) is correctly defined for $m = 0$ and $N_1 + N_2$ since the corresponding ψ interval is undefined for these two outcomes. In the following example (Δ_L, Δ_U) is defined to be $(-1,1)$ when $m = 0$ or $N_1 + N_2$.

Example 1.2: Again choose $N_1 = 2 = N_2$ and $\alpha = .01$. The conditional confidence intervals of McDonald et. al. (1974) are

x_1	x_2	Δ_L	Δ_U
0	2	-1	.4811
0	1	-1	.4808
1	2	-1	.4808
1	1	-.9316	.9316
0	0	-1	+1
2	2	-1	+1
2	1	-.4808	+1
1	0	-.4808	+1
2	0	-.4811	+1

When $p_1 = 3/4$ and $p_2 = 1/4$ then $\Delta = 1/2$, $\psi = 9$ and

$$P_{3/4,1/2}[\Delta_L(\underset{\sim}{X}) < 1/2 < \Delta_U(\underset{\sim}{X})] = 1 - P_{3/4,1/2}[(X_1,X_2) \in \{(0,2),(0,1)(1,2)\}]$$
$$= .949 < .99.$$

In Section 2 a method of constructing exact confidence intervals for Δ based on conditional ψ intervals will be proposed which satisfies the conditional confidence guarantee (1.6) and hence the weaker unconfidence guarantee (1.7). Section 3 discusses a second method for constructing Δ intervals which directly attempts to satisfy (1.7) rather than the conditional statement (1.6). Section 4 gives an example and draws some comparisons between the two methods while Section 5 summarizes the results and makes some recommendations regarding the use of Thomas-Gart (1977) intervals.

2. Conditional Confidence Intervals

For a given $\psi \in (0,\infty)$ there are infinitely many (p_1,p_2) pairs or equivalently (p_1,Δ) pairs associated with that ψ value. For each $\psi \in (0,\infty)$ let

$$(2.1)\quad D(\psi) \equiv \{\Delta \in (-1,1) \mid \exists\, p_1 \in I(\Delta) \ni p_1(1+\Delta-p_1)/(1-p_1)(p_1-\Delta) = \psi\}$$

be the set of differences associated with the odds ratio ψ.

The idea of the method is to use the set of all differences, Δ', associated with ψ's in an interval $(\psi_L(\underset{\sim}{X}), \psi_U(\underset{\sim}{X}))$ satisfying the conditional confidence guarantee (1.3) as the confidence interval for Δ. It will be shown below (Theorem 2.1) that the resulting confidence interval will also have conditional confidence level $(1-\alpha)$ and hence unconditional confidence level $(1-\alpha)$.

Formally the intervals are defined as follows. For any $(a,b) \subset (0,\infty)$ let

$$(2.2)\qquad E(a,b) \equiv \bigcup_{\psi\in(a,b)} D(\psi).$$

$E(a,b)$ is the set of all Δ's representable as $p_1 - p_2$ for some (p_1,p_2) satisfying $p_1(1-p_2)/(1-p_1)p_2 \in (a,b)$. $E(a,b)$ is always non-empty since the equation $\gamma = p_1(1-p_2)/(1-p_1)p_2$ always has solutions satisfying $0 < p_1,\ p_2 < 1$ for any $0 < \gamma < \infty$. The interval boundaries Δ_L and Δ_U are defined in terms of (ψ_L,ψ_U) and $E(\cdot,\cdot)$ as follows:

$$(2.3)\qquad \Delta_L(\underset{\sim}{X}) = \inf\ E(\psi_L,\psi_U) \quad \text{and}$$

(2.4) $$\Delta_U(\underset{\sim}{X}) = \sup E(\psi_L,\psi_U).$$

The following characterization of $E(a,b)$ will simplify the calculation of the sup and inf given in (2.3) and (2.4).

Lemma 2.1. $E(a,b) = \{\Delta = p_1 - p_1/((1-p_1)\psi + p_1) \mid (p_1,\psi) \in (0,1)\times(a,b)\}$.

Proof. If $\Delta \in E(a,b)$ then there exists $p_1 \in I(\Delta) \subset (0,1)$ so that $p_1(1+\Delta-p_1)/(1-p_1)(p_1-\Delta) = \psi$ for some ψ in (a,b). Solving for Δ shows Δ is in the right hand set above. Now suppose Δ is in the right hand set then there is $(p_1,\psi) \in (0,1)\times(a,b)$ satisfying $\Delta = p_1 - p_1/((1-p_1)\psi + p_1)$. It follows that $\Delta \in (-1,1)$ since $0 < p_1 < 1$ and $\psi > 0$. Solving for ψ gives $\psi = p_1(1+\Delta-p_1)/(1-p_1)(p_1-\Delta)$; it remains to show $p_1 \in I(\Delta)$ in order that $\Delta \in D(\psi) \subset E(a,b)$. Three cases are possible: (1) $-1 < \Delta < 0$, (2) $\Delta = 0$ and (3) $0 < \Delta < 1$. Only the first case will be considered as the remaining two are similar. It suffices to show $p_1 < 1+\Delta$ since $I(\Delta) = (0, 1+\Delta)$ when $-1 < \Delta < 0$. But $0 < \psi = p_1(1+\Delta-p_1)/(1-p_1)(p_1-\Delta) < \infty$ implies $0 < 1-p_1+\Delta$ since $\min\{p_1, 1-p_1, p_1-\Delta\} > 0$ when $-1 < \Delta < 0$ and the proof is completed.

Let $\Delta(p_1,\psi) \equiv p_1 - p_1/((1-p_1)\psi + p_1)$ and $R(a,b) \equiv (0,1)\times(a,b)$. From (2.3), (2.4) and Lemma 2.1 the $100(1-\alpha)\%$ conditional confidence limits $\Delta_L(\underset{\sim}{X})$ and $\Delta_U(\underset{\sim}{X})$ corresponding (ψ_L,ψ_U) are the solutions of the following optimization problems:

$$\Delta_L(\underset{\sim}{X}) = \inf_{R(\psi_L,\psi_U)} \Delta(p_1,\psi) \quad \text{and}$$

$$\Delta_U(\underset{\sim}{X}) = \sup_{R(\psi_L,\psi_U)} \Delta(p_1,\psi).$$

First a lemma describing the behavior of $\Delta(p_1,\psi)$ as a function p_1 for fixed ψ will be given.

Lemma 2.2.

(a) For any fixed $0 < \psi < 1$, $\Delta(p_1,\psi)$ is a (strictly) negative convex function in p_1; furthermore $\sup_{p_1\in(0,1)} \Delta(p_1,\psi) = 0$ and $\inf_{p_1\in(0,1)} \Delta(p_1,\psi)$

$$= \Delta\left(\frac{\sqrt{\psi}}{\sqrt{\psi}+1}, \psi\right) = \frac{\sqrt{\psi}-1}{\sqrt{\psi}+1}.$$

(b) For $\psi = 1$, $\Delta(p_1,1) \equiv 0$ for all $p_1 \in (0,1)$.

(c) For any fixed $1 < \psi < \infty$, $\Delta(p_1,\psi)$ is a (strictly) positive concave function in p_1; furthermore $\sup_{p_1\in(0,1)} \Delta(p_1,\psi) = \Delta\left(\frac{\sqrt{\psi}}{\sqrt{\psi}+1}, \psi\right) = \frac{\sqrt{\psi}-1}{\sqrt{\psi}+1}$ and $\inf_{p_1\in(0,1)} \Delta(p_1,\psi) = 0.$

Proof. Case (b) is immediate from the definition of $\Delta(p_1,\psi)$. It suffices to prove (a) since (c) follows from (a) and the easily verifiable relationship $\Delta(p_1,\psi) = -\Delta(1-p_1, 1/\psi)$. Fix $0 < \psi < 1$; for any $0 < p_1 < 1$ we have $(1-p_1)\psi + p_1 < 1$ and hence $\Delta(p_1,\psi) = p_1 - \frac{p_1}{(1-p_1)\psi + p_1} < p_1 - p_1 = 0.$ Taking derivatives of $\Delta(p_1,\psi)$ wrt p_1 gives

$$\frac{\partial\Delta}{\partial p_1} = 1 - \frac{\psi}{((1-p_1)\psi + p_1)^2} \quad \text{and} \tag{2.5}$$

$$\frac{\partial^2\Delta}{\partial p_1^2} = \frac{2\psi(1-\psi)}{(p_1(1-\psi)+\psi)^3}. \tag{2.6}$$

Now $\frac{\partial^2\Delta}{\partial p_1^2} > 0$ for $0 < p_1 < 1$ since $0 < \psi < 1$ and hence $\Delta(p_1,\psi)$ is

convex in p_1. The minimum of $\Delta(p_1,\psi)$ occurs at the solution of $\frac{\partial\Delta}{\partial p_1} = 0$ i.e., at $p_1 = \frac{\sqrt{\psi}}{\sqrt{\psi}+1}$ while $\lim_{p_1\to 0^+} \Delta(p_1,\psi) = \lim_{p_1\to 1^-} \Delta(p_1,\psi) = 0$ and hence the supremum of Δ over $(0,1)$ is 0. This completes the proof.

Figure A is a graph of $\Delta(p_1,\psi)$ vs. p_1 and ψ. The main result of the section will now be given.

Theorem 2.1. Suppose $(\psi_L(\underset{\sim}{X}), \psi_U(\underset{\sim}{X})) \subset (0,\infty)$ satisfies $P_{p_1,\Delta}[\psi_L(\underset{\sim}{X}) < \psi < \psi_U(\underset{\sim}{X})|m] \geq 1-\alpha \quad \forall -1 < \Delta < 1$ and $p_1 \in I(\Delta)$ for some integer m between zero and $N_1 + N_2$. The interval $(\Delta_L(\underset{\sim}{X}), \Delta_U(\underset{\sim}{X}))$ given by

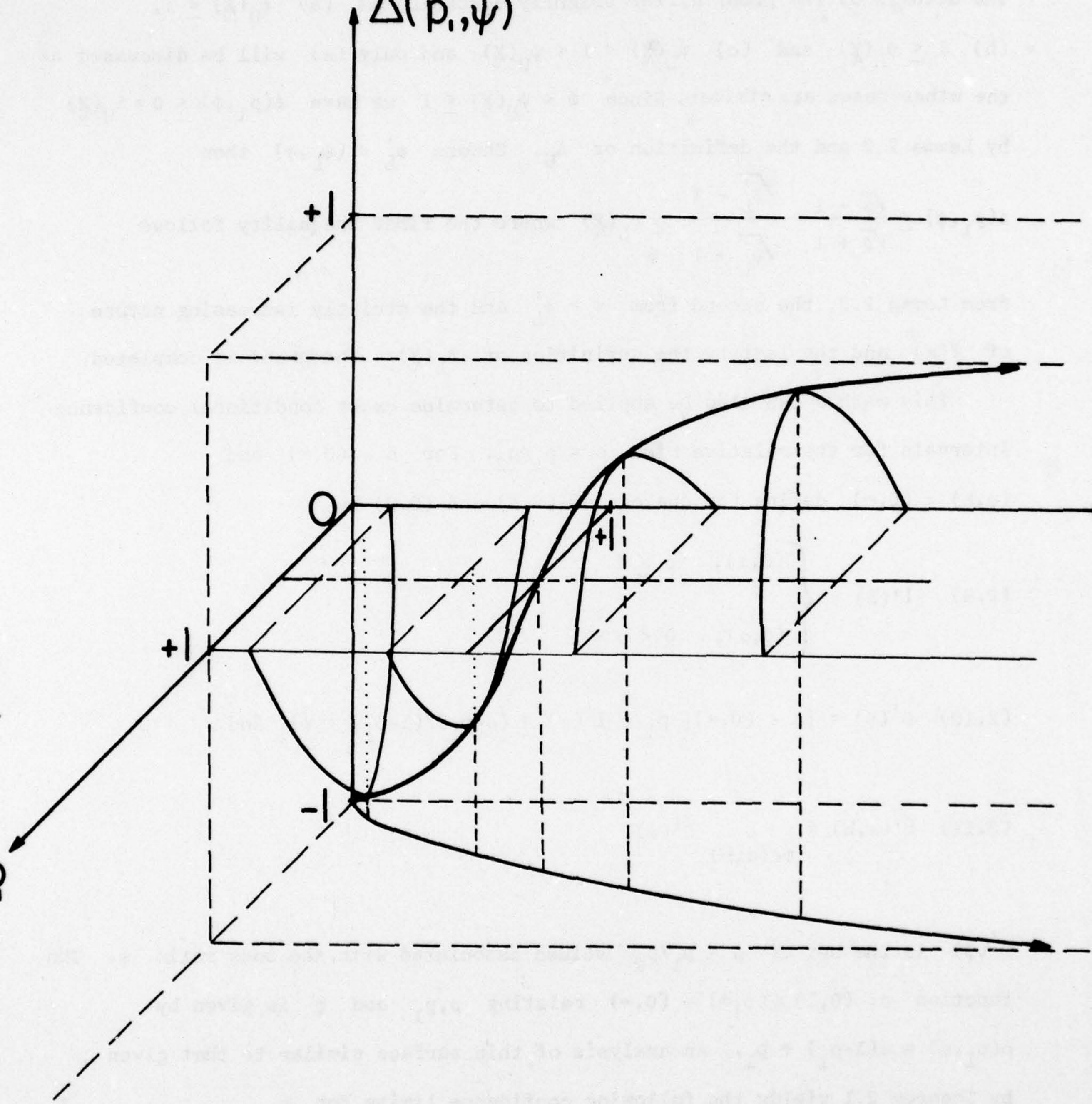
$$(2.7) \qquad \Delta_L(\underset{\sim}{X}) = \begin{cases} \dfrac{\sqrt{\psi_L(\underset{\sim}{X})}-1}{\sqrt{\psi_L(\underset{\sim}{X})}+1}, & 0 \leq \psi_L(\underset{\sim}{X}) < 1 \\[2ex] 0, & 1 \leq \psi_L(\underset{\sim}{X}) < \infty \end{cases}$$

$$(2.8) \qquad \Delta_U(\underset{\sim}{X}) = \begin{cases} 0, & 0 < \psi_U(\underset{\sim}{X}) < 1 \\[1ex] \dfrac{\sqrt{\psi_U(\underset{\sim}{X})}-1}{\sqrt{\psi_U(\underset{\sim}{X})}+1}, & 1 \leq \psi_U < \infty \\[2ex] +1, & \psi_U = \infty \end{cases}$$

satisfies $P_{p_1,\Delta}[\Delta_L(\underset{\sim}{X}) < \Delta < \Delta_U(\underset{\sim}{X})|m] \geq 1-\alpha \quad \forall -1 < \Delta < 1$ and $p_1 \in I(\Delta)$.

Proof. First note that $f(x) = (\sqrt{x}-1)/(\sqrt{x}+1)$ can be easily shown to be strictly increasing on the domain $(0,\infty)$. Now fix $\Delta \in (-1,1)$ and $p_1 \in I(\Delta)$ and let ψ be the corresponding odds ratio. Suppose the sample point $\omega \in [\Delta_L(\underset{\sim}{X}) < \psi < \psi_U(\underset{\sim}{X})]$; it suffices to prove $\omega \in [\Delta_L(\underset{\sim}{X}) < \Delta < \Delta_U(\underset{\sim}{X})]$.

A. THE SURFACE $\Delta(p_1,\psi)$ OVER $(0,1)\times(0,\infty)$.



The details of the proof differ slightly according as (a) $\psi_U(\underset{\sim}{X}) \leq 1$, (b) $1 \leq \psi_L(\underset{\sim}{X})$ and (c) $\psi_L(\underset{\sim}{X}) < 1 < \psi_U(\underset{\sim}{X})$ and only (a) will be discussed as the other cases are similar. Since $\psi < \psi_U(\underset{\sim}{X}) \leq 1$ we have $\Delta(p_1,\psi) < 0 = \Delta_U(\underset{\sim}{X})$ by Lemma 2.2 and the definition of Δ_U. Choose $\psi_L' \in (\psi_L,\psi)$ then $\Delta(p_1,\psi) \geq \dfrac{\sqrt{\psi} - 1}{\sqrt{\psi} + 1} > \dfrac{\sqrt{\psi_L'} - 1}{\sqrt{\psi_L'} + 1} \geq \Delta_L(\underset{\sim}{X})$ where the first inequality follows from Lemma 2.2, the second from $\psi > \psi_L'$ and the strictly increasing nature of $f(x)$ and the last by the definition of $\Delta_L(\underset{\sim}{X})$. The proof is completed.

This method can also be applied to determine exact conditional confidence intervals for the relative risk $\rho = p_1/p_2$. For $\rho \in (0,\infty)$ and $(a,b) \subset (0,\infty)$ define the analogs of (2.1) and (2.2) by

$$(2.9) \quad I'(\rho) = \begin{cases} (0,1), & \rho \geq 1 \\ (0,\rho), & 0 < \rho < 1 \end{cases}$$

$$(2.10) \quad D'(\psi) = \{\rho \in (0,\infty) | \exists p_1 \in I'(\rho) \ni (\rho - p_1)/(1-p_1) = \psi\} \text{ and}$$

$$(2.11) \quad E'(a,b) = \bigcup_{\psi\in(a,b)} D'(\psi).$$

$D'(\psi)$ is the set of $\rho = p_1/p_2$ values associated with the odds ratio ψ. The function ρ: $(0,1) \times (0,\infty) \to (0,\infty)$ relating ρ, p_1 and ψ is given by $\rho(p_1,\psi) = \psi(1-p_1) + p_1$. An analysis of this surface similar to that given by Theorem 2.1 yields the following confidence limits for ρ.

<u>Theorem 2.2</u>. Suppose $(\psi_L(\underset{\sim}{X}), \psi_U(\underset{\sim}{X}))$ satisfies the hypothesis of Theorem 2.1. The interval $(\rho_L(\underset{\sim}{X}), \rho_U(\underset{\sim}{X}))$ given by

$$\text{(2.12)} \qquad \rho_L(\underset{\sim}{X}) = \begin{cases} \psi_L(\underset{\sim}{X}), & 0 \le \psi_L(\underset{\sim}{X}) \le 1 \\ 1, & 1 < \psi_L(\underset{\sim}{X}) \end{cases}$$

$$\text{(2.13)} \qquad \rho_U(\underset{\sim}{X}) = \begin{cases} 1, & 0 < \psi_U(\underset{\sim}{X}) \le 1 \\ \psi_U(\underset{\sim}{X}), & 1 < \psi_L(\underset{\sim}{X}) \le \infty \end{cases}$$

satisfies $P_{p_1,\Delta}[\rho_L(\underset{\sim}{X}) < \rho < \rho_U(\underset{\sim}{X})|m] \ge 1-\alpha$ for all $\rho > 0$ and $p_1 \in I'(\rho)$.

The conditional method of this section is computationally simple to implement. The intervals are generally wider than the corresponding Gart intervals. Table 1 contains $100(1-\alpha)\%$ two-sided confidence intervals for $1 \le N_1,\ N_2 \le 10$ and $\alpha = .01,\ .05$ and $.10$. The computations were made on Cornell University's IBM 370/168 computer by applying (2.7) and (2.8) to the ψ intervals of Baptista and Pike (1977). Intervals are not listed for all possible (x_1,x_2) pairs. When $(x_1,x_2) = (0,0)$ or (N_1,N_2) the Δ interval is $(-1,+1)$ while for any other (x_1,x_2) not listed in Table I the Δ interval can be obtained from the relationships: $\Delta_L(x_1,x_2) = -\Delta_U(N_1-x_1,\ N_2-x_2)$ and $\Delta_U(x_1,x_2) = -\Delta_L(N_1-x_1,\ N_2-x_2)$.

In contrast to the conditional method of the present section the direct method of Section 3 is computationally difficult but generally produces tighter bounds than those in Table I.

3. Unconditional Confidence Intervals

3.1 Introduction

This section describes a method for constructing confidence intervals which directly satisfy (1.7). The following notation and definitions will be required throughout. For any $R \subset N_1 \times N_2$ and $\Delta \in (-1,1)$ let $\phi(R,\Delta)$ denote $\inf\{P_{p_1,\Delta}[(X_1,X_2) \in R] | p_1 \in I(\Delta)\}$ and $\Phi(R,\Delta)$ denote $\sup\{P_{p_1,\Delta}[(X_1,X_2) \in R] | p_1 \in I(\Delta)\}$.

Definition 3.1. A set $U \subset N_1 \times N_2$ is in the northwest corner (NWC) of $N_1 \times N_2$ provided (a) $(x_1,x_2) \in U \Rightarrow (x_1,\ell) \in U$ for $\ell = x_2+1,\ldots,N_2$ and (b) $(x_1,x_2) \in U \Rightarrow (\ell,x_2) \in U$ for $\ell = 0,\ldots,x_1-1$.

Conditions (a) and (b) are equivalent to requiring that the quadrant of points in $N_1 \times N_2$ to the "northwest" of any (x_1,x_2) in U also be in U.

Suppose $P = \{(d_0,d_1],(d_1,d_2],\ldots,[d_{n-1},d_n)\}$ is a partition of $(-1,1)$ satisfying $-1 = d_0 < d_1 < \ldots < d_n = 1$ and $S = \{R_1,\ldots,R_n\}$ is a collection of subsets of $N_1 \times N_2$. Here the events R_i need not be disjoint. For a fixed pair (P,S) define for each $(x_1,x_2) \in N_1 \times N_2$

$$(3.1)\quad T(x_1,x_2) = \bigcup_{j:d_j \le 0} (d_{j-1},d_j] \ \bigcup_{\substack{j:d_{j-1} \le 0,\\ d_j > 0}} (d_{j-1},d_j) \ \bigcup_{j:d_{j-1} > 0} [d_{j-1},d_j).$$

The first result is that $T(X_1,X_2)$ is a $100(1-\alpha)\%$ confidence region for Δ provided (P,S) satisfies

Condition 3.1. $\phi(R_j,\Delta) \ge 1-\alpha \ \forall\ \Delta \in [d_{j-1},d_j]$ and $\forall\ j \in \{1,\ldots,n\}$.

Lemma 3.1. Suppose (P,S) satisfies Condition 3.1 then

$P_{p_1,\Delta}[\Delta \in T(X_1,X_2)] \ge 1-\alpha$ for all $\Delta \in (-1,1)$ and all $p_1 \in I(\Delta)$.

<u>Proof</u>. Fix $\Delta \in (-1,1)$ and $p_1 \in I(\Delta)$; let i_0 be the index for which

$$\Delta \in \begin{cases} (d_{i_0-1}, d_{i_0}], & \text{if } d_{i_0} \leq 0 \\ (d_{i_0-1}, d_{i_0}), & \text{if } d_{i_0-1} \leq 0,\ d_{i_0} > 0 \\ [d_{i_0-1}, d_{i_0}), & \text{if } d_{i_0-1} > 0 \end{cases}.$$

Then $\Delta \in [d_{i_0-1}, d_{i_0}]$ and

$$P_{p_1,\Delta}[\Delta \in T(X_1,X_2)] = P_{p_1,\Delta}[(X_1,X_2) \in R_{i_0}]$$

$$\geq \phi(R_{i_0},\Delta) \text{ since } p_1 \in I(\Delta)$$

$$\geq 1-\alpha \text{ and the proof is completed.}$$

Clearly there are many (P,S) pairs satisfying Condition 3.1, for example, the trivial pair $P = \{(-1,1)\}$ and $S = \{N_1 \times N_2\}$; by Lemma 3.1 any of these can be used to generate $100(1-\alpha)\%$ confidence regions for Δ. Two additional intuitive requirements will be imposed on (P,S).

<u>Condition 3.2</u>. $T(x_1,x_2)$ must be an interval for all $(x_1,x_2) \in N_1 \times N_2$. This is equivalent to requiring that for each point $(x_1,x_2) \in N_1 \times N_2$ there are indices $f = f(x_1,x_2)$ and $\ell = \ell(x_1,x_2)$ satisfying $(x_1,x_2) \in R_i \iff f \leq i \leq \ell$.

The last requirement deals with the shape of R_i. First some additional notation must be introduced. Let $\Pi_1 = \Pi_1(R) \equiv \{i \mid \exists\, j \text{ with } (i,j) \in R\}$ and $\Pi_2 = \Pi_2(R) \equiv \{j \mid \exists\, i \text{ with } (i,j) \in R\}$ be the projections of $R \subset N_1 \times N_2$ onto the x_1 and x_2 axes respectively.

<u>Condition 3.3.</u> For R_ℓ (1) R_ℓ must be of the form $\mathcal{U}'_\ell - \mathcal{U}_\ell$ where $\mathcal{U}_\ell \subset \mathcal{U}'_\ell$ and both are NWC sets, (2) $\Pi_1(R_\ell)$ and $\Pi_2(R_\ell)$ must be intervals of integers and (3) there must exist real numbers α and $\beta > 0$ so that $\alpha + \beta i \geq s_i \equiv \min\{j|(i,j) \in R_\ell\}$ whenever $s_i > 0$, $\alpha + \beta i \leq \ell_i \equiv \max\{j|(i,j) \in R_\ell\}$ whenever $\ell_i < N_2$, $(j-\alpha)/\beta \geq S_j = \min\{i|(i,j) \in R_\ell\}$ whenever $S_j > 0$ and $(j-\alpha)/\beta \leq L_j = \max\{i|(i,j) \in R_\ell\}$ whenever $L_j < N_1$. Note that s_i, ℓ_i, S_j and L_j are only defined for $i \in \Pi_1(R_\ell)$ and $j \in \Pi_2(R_\ell)$.

Intuitively (1) and (2) are connectedness conditions which insure that R_ℓ has no holes while (3) is a technical condition under which $\phi(R_\ell,\Delta)$ is quasi-concave on $(-1,0]$ i.e. $\phi(R_\ell,\Delta_2) \geq \min\{\phi(R_\ell,\Delta_1),\phi(R_\ell,\Delta_3)\}$ for $-1 < \Delta_1 < \Delta_2 \leq \Delta_3 \leq 0$.

There are numerous criteria that can be employed to select from among those $(\mathcal{P},\mathcal{S})$ pairs satisfying Conditions (3.1)-(3.3). Two examples are (a) $(\mathcal{P},\mathcal{S})$ must minimize the average length of the intervals $T(x_1,x_2)$ and (b), a generalization of (a), $(\mathcal{P},\mathcal{S})$ must minimize a weighted sum of the lengths of $T(x_1,x_2)$. Both (a) and (b) are difficult to implement since they only indirectly stipulate conditions on $(\mathcal{P},\mathcal{S})$. This paper contains an algorithm for generating $(\mathcal{P},\mathcal{S})$ based on the so called "greedy" heuristic. It attempts to construct short intervals by forcing the R_i to be "small"; as few points as possible are added to R_i and as many points as possible are removed for R_i in order to construct R_{i+1}.

3.2 <u>The Algorithm</u>

Given $\alpha \in (0,1)$ and positive integers N_1 and N_2 the algorithm below generates a $(\mathcal{P},\mathcal{S})$ pair satisfying Conditions (3.1)-(3.3). Briefly,

Step 0 is an initialization procedure, Steps 1 through 5 form one inductive step and Step 6 generates the final (P,S) pair based on symmetry considerations. At each iteration Step 1 is entered with constants $-1 = d_0 < d_1 < \ldots < d_{i-1} < 0$ and regions $R_1,\ldots,R_i$ satisfying

(3.2) $\quad \phi(R_j,\Delta) \geq 1-\alpha \;\forall\, \Delta \in (d_{j-1},d_j] \quad \text{for } 1 \leq j \leq i-1$

(3.3) $\quad \phi(R_j,d_j) = 1-\alpha \quad \text{for } 1 \leq j \leq i-1$

(3.4) $\quad R_1,\ldots,R_i$ satisfy Condition 3.3

(3.5) $\quad$ if $I = I(i,x_1,x_2) = \{j \in \{1,\ldots,i\} | (x_1,x_2) \in R_j\}$ then $\bigcup_{j\in I} (d_{i-1},d_j]$ is either empty or an interval for every $(x_1,x_2) \in N_1 \times N_2$.

Two final pieces of notation are required. Given $R_j = \mathcal{U}_j' - \mathcal{U}_j$ as in (3.4) let $L_j \equiv N_1 \times N_2 - \mathcal{U}_j'$ be the set of points to the "southeast" of R_j; for arbitrary $S \subset N_1 \times N_2$ let $S^\rho = \{(x_1,x_2)|(N_1-x_1,N_2-x_2) \in S\}$ be the "rotation" of S and $|S|$ the cardinality of S.

Step 0. Set $d_0 = -1$ and $R_1 = \{(0,N_2)\}$. Construct $\mathcal{U}_0 \subset N_1 \times N_2$ so that

0.1 $\mathcal{U}_0$ is in the northwest corner of $N_1 \times N_2$ and $\mathcal{U}_0 \cap \mathcal{U}_0^\rho = \emptyset$

0.2 $N_1 \times N_2 - (\mathcal{U}_0 \cup \mathcal{U}_0^\rho)$ satisfies condition 3.3

0.3 $\Phi(\mathcal{U}_0 \cup \mathcal{U}_0^\rho,0) \leq \alpha$

0.4 if $B \subset N_1 \times N_2$ satisfies (0.1), (0.2) and (0.3) then either $|B| < |\mathcal{U}_0|$ or $|B| = |\mathcal{U}_0|$ and $\Phi(B \cup B^\rho,0) \geq \Phi(\mathcal{U}_0 \cup \mathcal{U}_0^\rho,0)$.

Set $L_0 = \mathcal{U}_0^\rho$, $i = 1$ and go to Step 1.

Step 1. Set $D_1 = \{\Delta \in (d_{i-1},1) | \phi(R_i,\Delta) < 1-\alpha\}$ and define $\Delta_1 = 1$ or $\inf D_1$ as $D_1 = \emptyset$ or $D_1 \neq \emptyset_1$.

If $(U_0 - U_i) \neq \emptyset$ go to Step 2.

If $(U_0 - U_i) = \emptyset$ set $d_i = \Delta_1$ and if $d_i < 0$ go to Step 4 while if $d_i \geq 0$ go to Step 6.

Step 2. Set $\bar{U} = \{(x_1,x_2) \in U_0 \cap R_i | R_i - \{(x_1,x_2)\}$ satisfies Condition 3.3 and $\phi(R_i - \{(x_1,x_2)\},\Delta) > 1-\alpha$ for some $\Delta \in (d_{i-1},\min\{\Delta_1,0\}]\}$.

If $\bar{U} \neq \emptyset$ go to Step 3.

If $\bar{U} = \emptyset$ set $d_i = \Delta_1$ and if $d_i < 0$ go to Step 4 while if $d_i \geq 0$ go to Step 6.

Step 3. For each $(x_1,x_2) \in \bar{U}$ set $d_i(x_1,x_2) = \inf\{\Delta \in [d_{i-1},\min\{\Delta_1,0\}] | \phi(R_i - \{(x_1,x_2)\},\Delta) > 1-\alpha\}$. Let $d_i = \min\{d_i(x_1,x_2) | (x_1,x_2) \in \bar{U}\}$. Construct $U^* \subset U_0 \cap R_i$ so that

3.1 $R_i - U^*$ satisfies Condition 3.3.

3.2 the infimum of $\{\Delta \in (d_{i-1},\min\{\Delta_1,0\}) | \phi(R_i - U^*,\Delta) > 1-\alpha\}$ exists and equals d_i

3.3 if $B \subset U_0 \cap R_i$ satisfies (3.1) and (3.2) then $|B| \leq |U^*|$.

Set $R_{i+1} = R_i - U^*$ and go to Step 1.

Step 4. Construct $S \subset L_i - L_0$ so that

4.1 $S \cup R_i$ satisfies Condition 3.3

4.2 $\inf\{\Delta \in (d_i,1) | \phi(S \cup R_i,\Delta) < 1-\alpha\} > d_i$ whenever the set is nonempty

4.3 if $B \subset L_i - L_0$ satisfies 4.1 and 4.2 then either $|B| > |S|$ or $|B| = |S|$ and $\phi(S \cup R_i,d_i) \geq \phi(B \cup R_i,d_i)$.

Set $\overline{R}_{i+1} = S \cup R_i$ and $\hat{U} = \{(x_1,x_2) \in U_0 \cap \overline{R}_{i+1} | \overline{R}_{i+1} - \{(x_1,x_2)\}$ satisfies Condition 3.3 and $\phi(\overline{R}_{i+1} - \{(x_1,x_2)\},d_i) \geq 1-\alpha\}$.

If $\hat{U} \neq \emptyset$ go to Step 5 while if $\hat{U} = \emptyset$ set $R_{i+1} = \overline{R}_{i+1}$ and go to Step 1.

Step 5. Construct $U^* \subset U_0 \cap \overline{R}_{i+1}$ so that (U^* possibly empty)

5.1 $\overline{R}_{i+1} - U^*$ satisfies Condition 3.3

5.2 $\inf\{\Delta \in (d_i,1) | \phi(\overline{R}_{i+1}-U^*,\Delta) < 1-\alpha\} > d_i$ whenever this set is nonempty

5.3 if $B \subset U_0 \cap \overline{R}_{i+1}$ satisfies (5.1) and (5.2) then either $|B| < |U^*|$ or $|B| = |U^*|$ and $\phi(\overline{R}_{i+1}-B,d_i) \leq \phi(\overline{R}_{i+1}-U^*,d_i)$.

Set $R_{i+1} = \overline{R}_{i+1} - U^*$ and go to Step 1.

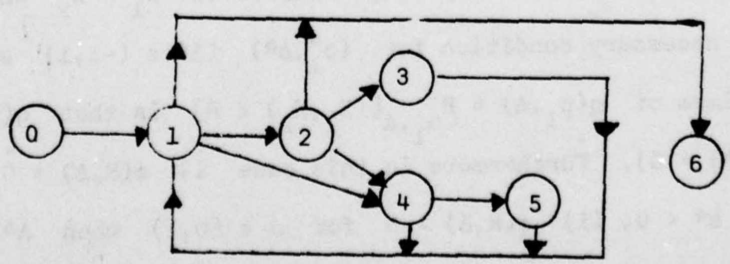
Step 6. If $R_i = R_i^\rho$ then complete (P,S) as follows: $P = \{(-1,d_1],\ldots,(d_{i-1},-d_{i-1}),[-d_{i-1},-d_{i-2}),\ldots,[-d_1,1)\}$ and $S = \{R_1,\ldots,R_{i-1},R_i,R_{i-1}^\rho,\ldots,R_1^\rho\}$.

If $R_i \neq R_i^\rho$ then complete (P,S) as follows: $P = \{(-1,d_1],\ldots,(d_{i-1},0],(0,-d_{i-1}],\ldots,[-d_1,1)\}$ and $S = \{R_1,\ldots,R_i,R_i^\rho,\ldots,R_1^\rho\}$.

3.3 Intuitive Description

The flow of the algorithm is diagrammed in Figure B. It begins in Step 0

B. FLOW CHART FOR ALGORITHM



by constructing two disjoint sets $\mathcal{U}_0$ and L_0 in the corners of the region $N_1 \times N_2$ having total probability less than α when $\Delta = 0$ and for any p_1. $\mathcal{U}_0$ and L_0 are used in the completion process of Step 6. Step 0 also initializes $R_1 = \{(0,N_2)\}$ and $d_0 = -1$. Next a trial d_i, Δ_1, is constructed in Step 1 so that $\phi(R_i,\Delta) \geq 1-\alpha$ for all $\Delta \in (d_{i-1},\Delta_1]$. Then in Steps 2 and 3 the algorithm checks whether any points, $\mathcal{U}^*$, can be deleted from R_i if d_i is permitted to be smaller than Δ_1. If so, a revised d_i is constructed and then R_{i+1} is generated by deleting $\mathcal{U}^*$ from R_i; Step 1 is reentered. If no points can be removed from R_i and $d_i < 0$ then the algorithm goes to Step 4 and constructs a trial R_{i+1} by adding points, S, to R_i so that $\phi(R_{i+1} = R_i \cup S, d_i) \geq 1-\alpha$. Then in the remainder of Step 4 and in Step 5 it checks whether any other points, $\mathcal{U}^*$, can be deleted from $S \cup R_i$ while keeping $\phi(S \cup R_i - \mathcal{U}^*, d_i) \geq 1-\alpha$. At the stage when $d_i \geq 0$ the remaining sets R_j and points d_j are constructed via symmetry considerations in Step 6.

3.4 Properties of the Algorithm

The proofs that $(\mathcal{P},\mathcal{S})$ satisfies Conditions (3.1) to (3.3) will be given in this subsection and the appendix. Some preliminary results regarding the behavior of $\Phi(R,\Delta)$ and $\phi(R,\Delta)$ will be stated first. Let $c\ell(\mathcal{I}(\Delta^*))$ denote the closure of $\mathcal{I}(\Delta^*)$.

Lemma 3.2. Suppose R is a nonempty proper subset of $N_1 \times N_2$ which satisfies Condition 3.3. A necessary condition for (p_1^*,Δ^*) $(\Delta^* \in (-1,1)$ and $p_1^* \in c\ell(\mathcal{I}(\Delta^*))$ to be a local minimum of $q(p_1,\Delta) = P_{p_1,\Delta}[(X_1,X_2) \in R]$ is that $q(p_1^*,\Delta^*) = 0$ (and hence $\phi(R,\Delta^*) = 0$). Furthermore in this case i) $\phi(R,\Delta) = 0$ for $\Delta \in (-1,0]$ when $\Delta^* < 0$, ii) $\phi(R,\Delta) = 0$ for $\Delta \in [0,1)$ when $\Delta^* > 0$, and

iii) either $\phi(R,\Delta) = 0$ for $\Delta \in (-1,0]$ or $\Delta \in [0,1)$ when $\Delta^* = 0$.

See the appendix for the proof.

Lemma 3.3. Suppose R satisfies the conditions of Lemma 3.2. Then $\phi(R,\Delta)$, regarded as a function of Δ, is quasiconcave on $(-1,0]$, i.e. if $-1 < \Delta_1 < \Delta_2 < \Delta_3 \leq 0$ then $\phi(R,\Delta_2) \geq \min\{\phi(R,\Delta_1),\phi(R,\Delta_3)\}$.

Proof. Suppose the contrary, then there exist $-1 < \Delta_1 < \Delta_2 < \Delta_3 \leq 0$ for which $\phi(R,\Delta_1) > \phi(R,\Delta_2)$ and $\phi(R,\Delta_3) > \phi(R,\Delta_2)$. It can be shown that $\phi(R,\Delta)$ is continuous in Δ and hence there exists $\Delta^* \in (\Delta_1,\Delta_3)$ satisfying $\phi(R,\Delta^*) = \min_{\Delta\in[\Delta_1,\Delta_3]} \phi(R,\Delta)$. Choose $p^* \in c\ell(I(\Delta^*))$ so that $P_{p^*,\Delta^*}[(X_1,X_2) \in R] = \phi(R,\Delta^*)$. Two cases arise: $\phi(R,\Delta^*) = 0$ and $\phi(R,\Delta^*) > 0$. If $\phi(R,\Delta^*) = P_{p^*,\Delta^*}[(X_1,X_2) \in R] = 0$ then $p^* = 0$ or $1 + \Delta^*$ since $R \neq \emptyset$. It follows from Lemma 3.2 that $\phi(R,\Delta) = 0$ for all $\Delta \in (-1,0]$ which is a contradiction. If $P_{p^*,\Delta^*}[(X_1,X_2) \in R] > 0$ then choose any ε-ball, B, about (p^*,Δ^*) so that $B \cap \{(p_1,\Delta)|\Delta \in (-1,1)$ and $p_1 \in I(\Delta)\} \subset \{(p_1,\Delta)|\Delta \in (\Delta_1,\Delta_3)$ and $p_1 \in I(\Delta)\}$. By Lemma 3.2 (p^*,Δ^*) is not a local minimum and hence there exist $(\bar{p},\bar{\Delta})$ in $B \cap \{(p_1,\Delta)|\Delta \in (-1,1)$ and $p_1 \in I(\Delta)\}$ so that $P_{\bar{p},\bar{\Delta}}[(X_1,X_2) \in R] < P_{p^*,\Delta^*}[(X_1,X_2) \in R]$. This implies that $\phi(R,\bar{\Delta}) < \phi(R,\Delta^*)$ which is a contradiction and the proof is complete.

The following lemma will be used to show that the extension of Step 6 gives regions which satisfy Condition 3.1.

Lemma 3.4. Fix $R \subset N_1 \times N_2$ and let $R^\rho = \{(x_1,x_2)|(N_1-x_1,N_2-x_2) \in R\}$ be the rotation of R. Then $\Phi(R,\Delta) = \Phi(R^\rho,-\Delta)$ for all $\Delta \in (-1,1)$ and $\phi(R,\Delta) = \phi(R^\rho,-\Delta)$ for all $\Delta \in (-1,1)$.

Proof. It suffices to consider $\phi(R,\Delta)$ since $\Phi(R,\Delta) = 1 - \phi(R^c,\Delta)$ where R^c is the complement of R in $N_1 \times N_2$. Fix $\Delta \in (-1,1)$; it is easy to check

that $p_1 \in c\ell(I(\Delta)) \Longleftrightarrow 1-p_1 \in c\ell(I(-\Delta))$. Then it follows that

$$P_{p_1,\Delta}[(X_1,X_2) \in R] = \sum_{(i,j)\in R} \binom{N_1}{i}\binom{N_2}{j} p_1^i (1-p_1)^{N_1-i} (p_1-\Delta)^j (1-p_1+\Delta)^{N_2-j}$$

$$= \sum_{(\ell,k)\in R^\rho} \binom{N_1}{\ell}\binom{N_2}{k} (1-p_1)^\ell p_1^{N_1-\ell} (1-p_1+\Delta)^k (p_1-\Delta)^{N_2-k}$$

from the change of variables $\ell = N_1-i$ and $k = N_2-j$

$$= P_{1-p_1,-\Delta}[(X_1,X_2) \in R^\rho].$$

So if $p_1^* \in c\ell(I(\Delta))$ satisfies $\phi(R,\Delta) = P_{p_1^*,\Delta}[(X_1,X_2) \in R]$ then $\phi(R,\Delta^*) = P_{1-p_1^*,-\Delta}[(X_1,X_2) \in R^\rho] \geq \phi(R^\rho,-\Delta)$. A similar argument gives the reverse inequality and completes the proof.

It is easy to check that $\phi(R_1 = \{(0,N_2)\},\Delta) = |\Delta|^m$ on $(-1,0]$ where $m = \max\{N_1,N_2\}$; hence the first time Step 1 is entered Δ_1 will be $-(1-\alpha)^{\frac{1}{m}}$ and the algorithm will go to Step 4. It will now be shown that the algorithm is executable at all later iterations since Step 4 can always be implemented.

<u>Lemma 3.5</u>. Suppose $\{R_1,\ldots,R_i\}$ and $\{d_0,\ldots,d_{i-1}\}$ satisfy Conditions 3.1 and 3.3. If Step 4 is entered then there exists a nonempty set, S, satisfying (4.1) and (4.2) of Step 4.

<u>Proof</u>. Fix $S = L_i - L_0$. We claim $S \neq \emptyset$ under the hypotheses of the lemma. Step 4 is entered only if $D_1 = \{\Delta \in (d_{i-1},0) | \phi(R_i,\Delta) < 1-\alpha\}$ is nonempty and $d_i \equiv \inf D_1 < 0$. Hence there exists $\overline{\Delta} \in (d_i,0)$ such that $\phi(R_i,\overline{\Delta}) < 1-\alpha$ while $\phi(R_i,d_{i-1}) \geq 1-\alpha$ by construction. The level set

$\{\Delta \in (-1,0) | \phi(R_i,\Delta) \geq 1-\alpha\}$ is convex since ϕ is quasiconcave and so $\phi(R_i,0) < 1-\alpha$. On the other hand if $S = L_i - L_0 = \emptyset$ then $L_i = L_0$ and $\phi(R_i,0) = \phi(N_1 \times N_2 - (L_0 \cup \mathcal{U}_i),0) \geq \phi(N_1 \times N_2 - (L_0 \cup \mathcal{U}_0),0) \geq 1-\alpha$ where the first inequality follows from $\mathcal{U}_i \subset \mathcal{U}_0$ and the second from Step 0. This contradiction shows $S \neq \emptyset$. It is proved in the appendix that $S \cup R_i = N_1 \times N_2 - (L_0 \cup \mathcal{U}_i)$ satisfies (4.1). The above inequalities show that $\phi(S \cup R_i,0) \geq 1-\alpha$ while $\phi(S \cup R_i,d_i) \geq \phi(R_i,d_i) \geq 1-\alpha$. Hence $\{\Delta \in (d_i,1) | \phi(S \cup R_i,\Delta) < 1-\alpha\} \subset (0,1)$ by the convexity of $\{\Delta \in (-1,0] | \phi(S \cup R_i,\Delta) \geq 1-\alpha\}$. This implies that $(S \cup R_i)$ satisfies (4.2) and completes the proof.

<u>Theorem 3.1</u>. The $(\mathcal{P},\mathcal{S})$ pair constructed by the algorithm satisfies Condition 3.1.

<u>Proof</u>. The inequality $\phi(R_i,\Delta) \geq 1-\alpha$ for $\Delta \in [d_{i-1},d_i]$ holds by construction for any i with $d_i \leq 0$. Lemma 3.5 shows that the inductive steps 1 through 5 can stop only with $d_i \geq 0$ and the execution of Step 6. Lemma 3.4 shows that the above inequality holds for i with $d_i \geq 0$ and the result is proved.

The next two results demonstrate Condition 3.2 that $T(x_1,x_2)$ is always an interval. They are based on the following easily derivable representation of $T(x_1,x_2)$.

$$T(x_1,x_2) = T_1(x_1,x_2) \cup (-T_1(N_1-x_1,N_2-x_2))$$

where $T_1(x_1,x_2) = (-1,0] \cap \bigcup_{j \in I(x_1,x_2)} (d_j,d_{j+1}]$, $I(x_1,x_2) = \{j | (x_1,x_2) \in R_{j+1}, d_j < 0\}$ and $-T_1(x_1,x_2) = [0,1) \cap \bigcup_{j \in I(x_1,x_2)} [-d_{j+1},-d_j)$. The set $T_1(x_1,x_2)$

is that part of $T(x_1,x_2)$ on and to the left of the origin; the set $(-T,(N_1-x_1,N_2-x_2))$ is that part of $T(x_1,x_2)$ on and to the right of the origin.

Lemma 3.7. For every $(x_1,x_2) \in N_1 \times N_2$, $T_1(x_1,x_2)$ is an interval.

Proof. If $T_1(x_1,x_2) = \emptyset$ the result is trivial; assume $T_1(x_1,x_2) \neq \emptyset$. Suppose $T_1(x_1,x_2)$ is not an interval so that there exists Δ_1, Δ_2 and Δ_3 satisfying $-1 < \Delta_1 < \Delta_2 < \Delta_3 \leq 0$ for which $\Delta_1,\Delta_3 \in T_1(x_1,x_2)$ and $\Delta_2 \notin T_1(x_1,x_2)$. Let $i_1 < i_2 < i_3$ be the indices for which $\Delta_1 \in (d_{i_1-1},d_{i_1}]$, $\Delta_2 \in (d_{i_2-1},d_{i_2}]$ and $\Delta_3 \in (d_{i_3-1},d_{i_3}]$. The point $(x_1,x_2) \in R_{i_1} = \mathcal{U}'_{i_1} - \mathcal{U}_{i_1} \Rightarrow (x_1,x_2) \in \mathcal{U}'_{i_2} \subset \mathcal{U}'_{i_3}$ since the sequence $\{\mathcal{U}'_j\}$ is nondecreasing by construction. But $(x_1,x_2) \notin R_{i_2}$ and $(x_1,x_2) \in R_{i_3} \Rightarrow (x_1,x_2) \in \mathcal{U}_{i_2} - \mathcal{U}_{i_3}$ which is impossible since the $\{\mathcal{U}_j\}$ sequence is also nondecreasing and completes the proof.

Theorem 3.2. For every $(x_1,x_2) \in N_1 \times N_2$, $T(x_1,x_2)$ satisfies Condition 3.2.

Proof. If $T_1(x_1,x_2) = \emptyset$ or $T_1(N_1-x_1,N_2-x_2) = \emptyset$ then the result is immediate from Lemma 3.7. Now suppose $T_1(x_1,x_2) \neq \emptyset$, $T_1(N_1-x_1,N_2-x_2) \neq \emptyset$ and $T(x_1,x_2)$ is not connected. Let τ be the index for which $0 \in (d_\tau,d_{\tau+1}]$. It follows that either $0 \notin T_1(x_1,x_2)$ and/or $0 \notin T_1(N_1-x_1,N_2-x_2)$ $\Leftrightarrow$ either $(x_1,x_2) \in \mathcal{U}_{\tau+1}$ and/or $(N_1-x_1,N_2-x_2) \in \mathcal{U}_{\tau+1}$. The last equivalence follows from the fact that if $(x_1,x_2) \in L_{\tau+1} \Rightarrow (x_1,x_2) \in L_j$ for $j \leq \tau$ since $\{L_i\}$ is a nonincreasing sequence $\Rightarrow T_1(x_1,x_2) = \emptyset$ which is impossible. Assume wlog that $(x_1,x_2) \in \mathcal{U}_{\tau+1} \subset \mathcal{U}_0 = L_0^\rho \subset L_{\tau+1}^\rho \Rightarrow (N_1-x_1,N_2-x_2) \in L_{\tau+1} \Rightarrow$ $T_1(N_1-x_1,N_2-x_2) = \emptyset$ which is a contradiction. A similar contradiction arises if $(N_1-x_1,N_2-x_2) \in \mathcal{U}_{\tau+1}$ and completes the proof.

Theorem 3.3. All regions R_i in the family S constructed by the algorithm satisfy Condition 3.3.

Proof. The regions $R_1,\ldots,R_{\tau+1}(0 \in (d_\tau, d_{\tau+1}])$ satisfy Condition 3.3 by construction. It can easily be checked that if R satisfies Condition 3.3 then R^ρ also does. The result follows since for $i > \tau$, $R_i = R_j^\rho$ for some $j \le \tau+1$.

4. An Example and Conclusions

This section will present a detailed example to illustrate the unconditional method of Section 3 and make some comparisons among the Thomas-Gart (TG) method, the conditional method of Section 2 and the unconditional method of Section 3.

The example below constructs 80% confidence intervals when $N_1 = N_2 = 2$. The sample size $N_1 = N_2 = 2$ was chosen to keep the computations feasible by hand while the relatively large value of $\alpha = .2$ was selected to illustrate Step 5. Step 5 is not used for $\alpha < .125$.

Initialization:

Step 0: Set $d_0 = 1$, $R_1 = \{(0,2)\}$. Let $\mathcal{U} = \{(0,2)\}$; $\mathcal{U}$ satisfies 0.1, 0.2 and 0.3 ($\Phi(\mathcal{U} \cup \mathcal{U}^\rho, 0) = .125 < .2$). $\mathcal{U}$ also satisfies 0.4 since for any other candidate set B either $\mathcal{U}_1 \equiv \{(0,2),(0,1)\} \subset B$ or $\mathcal{U}_2 \equiv \{(0,2),(1,2)\} \subset B$ and hence $\Phi(B \cup B^\rho) \geq \Phi(\mathcal{U}_1 \cup \mathcal{U}_1^\rho, 0) = \Phi(\mathcal{U}_2 \cup \mathcal{U}_2^\rho, 0)$ $= .375 > .2$. So $\mathcal{U}_0 = \mathcal{U} = \{(0,2)\}$ and $L_0 = \{(2,0)\}$.

Iteration 1:

Step 1: $\phi(R_1,\Delta) = \Delta^2$ or 0 according as $\Delta < 0$ or $\Delta \geq 0$ so $D_1 = \{\Delta \in (-1,1) | \phi(R_1,\Delta) < .8\} = (-\sqrt{.8},1)$ and $\Delta_1 = \inf D_1 = -\sqrt{.8}$ $= -.8944$. Go to Step 2.

Step 2: $R_1 - \{(0,2)\} = \phi \Rightarrow \sup\limits_{\Delta \in (-1,-.8944)} \phi(R_1 - \{(0,2)\},\Delta) = 0 < .8 \Rightarrow \overline{\mathcal{U}} = \phi$. Set $d_1 = -.8944$ and go to Step 4.

Step 3. Candidate sets satisfying (4.1) and $|S| = 1$ are $S_1 = \{(0,1)\}$ and $S_2 = \{(1,2)\}$. In both cases $\phi(S_i \cup R_1,\Delta) = \phi(R_1,\Delta)$ implies that

$\inf\{\Delta \in (-.8944,1) | \phi(S_i \cup R_1,\Delta) < .8\} = -.8944$ violating 4.2. Candidate sets satisfying (4.1) and $|S| = 2$ are $S_1 = \{(0,0),(0,1)\}$, $S_2 = \{(1,2),(2,2)\}$ and $S_3 = \{(0,1),(1,2)\}$. For i=1 and 2 $\phi(S_i \cup R_1,\Delta) = \phi(R_1,\Delta) \Rightarrow \inf\{\Delta \in (-.8944,1) | \phi(S_i \cup R_1,\Delta) < .8\} = -.8944$ again violating 4.2. However

$$\phi(S_3 \cup R_1,\Delta) = \begin{cases} (1-\Delta)^3(5+3\Delta)/16, & 0 < \Delta < -.4415 \\ -2\Delta - \Delta^2, & -.4415 \leq \Delta < 0 \\ 0, & 0 \leq \Delta < 1 \end{cases}$$

$\Rightarrow \inf\{\Delta \in (-.8944,1) | \phi(S_3 \cup R_1,\Delta) < .8\} = -.5754 > d_1$. By construction S_3 satisfies 4.3 and hence $S = S_3$. Set $\overline{R}_2 = \{(0,2),(0,1),(1,2)\}$; $\hat{U} = \phi$ since $\phi(R_2 - \{(0,2)\},-.8944) = .1795$. Finally set $R_2 = \overline{R}_2$ and go to Step 1.

Iterations 2, 3, 4 and part of 5 are summarized in Table 2. The values of Δ_1 listed in column 3 are calculated in Step 1. In all 4 cases the maximum of $\phi(R_i - U_0,\Delta)$ over Δ in $[d_{i-1},\Delta_1]$ is less than .8 implying that $\overline{U} = \phi$ and the algorithm goes from Step 2 to Step 4. Column 7 shows that $\hat{U} = \phi$ in iterations 2, 3, and 4 while $\hat{U} = \{(0,2)\}$ in iteration 5. We now complete iteration 5.

<u>Iteration 5</u> (cont.):

<u>Step 5</u>: Set $U^* = \{(0,2)\}$; $\overline{R}_6 - U^* = \{(0,0),(0,1),(1,0),(1,1),(1,2),(2,1),(2,2)\}$ satisfies (5.1); (5.2) holds since $\inf\{\Delta \in (-.0757,1) | \phi(\overline{R}_6 - U^*,\Delta) < .8\} = .3137 > -.0757$ while (5.3) is trivially satisfied. Set $R_6 = \overline{R}_6 - \{(0,2)\}$ and go to Step 1.

2. ITERATIONS 2-5 OF ALGORITHM

Iter.	R_i	Δ_1/d_i	$\max\limits_{[d_{i-1},\Delta_1]} \phi(R_i-\mathcal{U}_0,\Delta)$	Sets S of 4.1	inf of 4.2	$\phi(\bar{R}_{i+1}-\mathcal{U}_0, d_i)$
2	{(0,2),(0,1),(1,2)}	$\Delta_1 = -.5754$ $d_2 = -.5754$	.4150	{(0,0)} {(2,2)} {(1,1)}[1]	-.5754 -.5754 -.5528	.4886
3	{(0,2),(0,1),(1,2) (1,1)}	$\Delta = -.5528$ $d_3 = -.5528$	.4944	{(0,0)} {(2,2)} {(0,0),(1,0)} {(2,1),(2,2)} {(0,0),(2,2)}[1]	-.5528 -.5528 -.5528 -.5528 -.1649	.5994
4	{(0,2),(0,1),(1,2), (1,1),(0,0),(2,2)}	$\Delta_1 = -.1649$ $d_4 = -.1649$	.6962	{(2,1)}[2] {(1,0)}[2]	-.0757 -.0757	.7481
5	{(0,2),(0,1),(1,2), (1,1),(0,0),(2,2), (1,0)}	$\Delta_1 = -.0757$ $d_5 = -.0757$	.7481	{(2,1)}[1]	1	.8707[3]

[1] The set S chosen in Step 4.

[2] Both candidate sets having $|S| = 1$ satisfy 4.1 and 4.2; $\phi(R_4 \cup \{(2,1)\}, d_4) = \phi(R_4 \cup \{(1,0)\}, d_4)$ and so 4.3 fails to select one of the two. $S = \{(1,0)\}$ is chosen arbitrarily.

[3] $\hat{\mathcal{U}} \neq \phi$ and so Step 5 must be executed during Iteration 5.

Iteration 6:

Step 1: $D_1 = (.3137,1)$ so that $\Delta_1 = .3137$; $\mathcal{U}_0 - \mathcal{U}_6 = \phi$ so set $d_6 = .3137$ and go to Step 6.

Step 6:

P	S
(-1,-.8944]	$R_1 = \{(0,2)\}$
(-.8944,-.5754]	$R_2 = \{(0,2),(0,1),(1,2)\}$
(-.5754,-.5528]	$R_3 = \{(0,2),(0,1),(1,2),(1,1)\}$
(-.5528,-.1649]	$R_4 = \{(0,2),(0,1),(1,2),(1,1),(0,0),(2,2)\}$
(-.1649,-.0757]	$R_5 = \{(0,2),(0,1),(1,2),(1,1),(0,0),(2,2),(1,0)\}$
(-.0757, .0757]	$R_6 = R_6^\rho = \{(0,1),(1,2),(1,1),(0,0),(2,2),(1,0),(2,1)\}$
[.0757,.1649)	$R_7 = R_5^\rho = \{(2,0),(2,1),(1,0),(1,1),(2,2),(0,0),(1,2)\}$
[.1649,.5528)	$R_8 = R_4^\rho = \{(2,0),(2,1),(1,0),(1,1),(2,2),(0,0)\}$
[.5528,.5754)	$R_9 = R_3^\rho = \{(2,0),(2,1),(1,0),(1,1)\}$
[.5754,.8944)	$R_{10} = R_2^\rho = \{(2,0),(2,1),(1,0)\}$
[.8944,1)	$R_{11} = R_1^\rho = \{(2,0)\}$

The 80% confidence intervals for $\Delta = \rho_1 - \rho_2$ are:

x_1	x_2	$T(x_1,x_2)$
0	2	(-1,-.0757]
0	1	(-.8944,.0757)
1	2	(-.8944,.1649)
1	1	(-.5754,.5754)
0	0	(-.5528,.5528)
2	2	(-.5528,.5528)
2	1	(-.0757,.8944)
1	0	(-.1649,.8944)
2	0	[.0757,1) .

<u>Remark 4.1</u>. As iteration 4 illustrates, there can be several sets S satisfying (4.1)-(4.3) and several sets $\mathcal{U}^*$ satisfying (3.1)-(3.3) or (5.1)-(5.3). Randomization or an arbitrary selection rule can be used to break such ties. For example the following rule is used here: "choose the set S minimizing $\frac{1}{|S|}\sum_{(i,j)\in S} i$; randomize among sets tied according to this criteria".

<u>Remark 4.2</u>. So far no mention has been made of the computational work required to implement the algorithm. In the example when $R_2 = \{(0,1),(0,2),(1,2)\}$ and $\Delta < 0$:

$$\phi(R_2,\Delta) = \min_{p_1\in[0,1+\Delta]} P_{p_1,\Delta}[(X_1,X_2) \in R_2]$$

$$= \min_{p_1\in[0,1+\Delta]} \{-3p_1^4 + 6(\Delta+1)p_1^3 - (3\Delta^2 + 10\Delta + 5)p_1^2 + 2(2\Delta^2 + 3\Delta + 1)p_1 - (\Delta^2+2\Delta)\}.$$

Minimizing $P_{p_1,\Delta}[(X_1,X_2) \in R_2]$ in p_1 for fixed Δ requires comparison of the function values at the bounding points 0 and $1+\Delta$ and at the zeroes of the equation $P'_{p_1,\Delta}[(X_1,X_2) \in R_2] = 0$ where the prime denotes partial differentiation with respect to p_1. In this case the zeroes of a $3^{\underline{d}}$ degree polynomial must be computed. In the general case the zeroes of an (N_1+N_2-1) degree polynomial must be computed. This particular case can be simplified by reparameterizing the problem to $w = (1+\Delta)/2 - p_1$.

For $\Delta < 0$:

$$\phi(R_2,\Delta) = \min_{|w|\le\frac{1+\Delta}{2}} P_{w+\frac{1+\Delta}{2},\Delta} [X_1,X_2) \in R_2]$$

$$= \min_{|w|\le\frac{1+\Delta}{2}} \{(1-\Delta)^4/16 + (1-\Delta^2)(1-\Delta)^2/4 + (1.5\Delta^2 - \Delta - .5)w^2 - 3w^4\}.$$

The terms in brackets are a function of $t = w^2$, say, $g(w^2)$. So

$$\phi(R_2,\Delta) = \min_{0<t<\frac{(1+\Delta)^2}{4}} \{g(0) + (1.5\Delta^2 - \Delta - .5)t - 3t^2\}$$

$$= \min_{0\leq t\leq\frac{(1+\Delta)^2}{4}} g(t).$$

Clearly $g(t)$ is concave; its minimum is either achieved at 0 or $(1+\Delta)^2/4$. For $\Delta \leq -.4415$, $\phi(R_2,\Delta) = g(0) = (1-\Delta)^4/16 + (1-\Delta)^2(1-\Delta^2)/4$; for $-.4415 < \Delta \leq 0$, $\phi(R_2,\Delta) = g((1+\Delta)^2/4) = -\Delta(2+\Delta)$.

In general, the order of the polynomial in p_1, $P_{p_1,\Delta}[(X_1,X_2) \in R_\ell]$, can be halved by the same reparameterization whenever $N_1 = N_2$ and $R_\ell = \{(N_1-x_2, N_2-x_1)|(x_1,x_2) \in R_\ell\}$. These conditions imply that $P_{p_1,\Delta}[(X_1,X_2) \in R_\ell]$ is symmetric in p_1 about $(1+\Delta)/2$.

When the algorithm is applied to the $N_1 = N_2 = 2$ case for $\alpha = .01$, .05 and .1 it yields the Δ intervals of Table 3. The corresponding conditional Δ intervals of Section 2 are listed in Table 4 and the Thomas-Gart Δ intervals based on the Baptista-Pike ψ intervals are listed in Table 5. Note that the 99% Thomas-Gart Δ intervals of Example 1.2 are based on the Thomas (1971) ψ intervals. Hence the intervals of Table 5 are never wider than those listed in Example 1.2. We shall make several comparisons among these intervals.

We begin by continuing Example 1.2. The actual coverage probabilities of $\Delta = 1/2$ are listed below when $p_1 = 3/4$ and $p_2 = 1/4$. The intervals are taken from Tables 3,4 and 5; all have nominal 99% confidence coefficients.

3. UNCONDITIONAL CONFIDENCE INTERVALS FOR $N_1 = N_2 = 2$

x_1	x_2	90%		95%		99%	
0	2	-1	0	-1	.0543	-1	.3676
0	1	-.9487	.2747	-.9747	.4294	-.9950	.6708
1	2	-.9487	.3591	-.9747	.5028	-.9950	.7183
1	1	-.7147	.7147	-.8048	.8048	-.9160	.9160
0	0	-.6838	.6838	-.7764	.7764	-.9000	.9000
2	2	-.6838	.6838	-.7764	.7764	-.9000	.9000
2	1	-.2747	.9487	-.4294	.9749	-.6708	.9950
1	0	-.3591	.9487	-.5028	.9747	-.7183	.9950
2	0	0	1	-.0543	1	-.3676	1

4. CONDITIONAL CONFIDENCE INTERVALS (SECTION 2) FOR $N_1 = N_2 = 2$

x_1	x_2	90%		95%		99%	
0	2	-1	.1178	-1	.2515	-1	.4811
0	1	-1	.5	-1	.6268	-1	.8174
1	2	-1	.5	-1	.6268	-1	.8174
1	1	-.7151	.7151	-.7945	.7945	-.9043	.9043
0	0	-1	1	-1	1	-1	1
2	2	-1	1	-1	1	-1	1
2	1	-.5	1	-.6268	1	-.8174	1
1	0	-.5	1	-.6268	1	-.8174	1
2	0	-.1178	1	-.2515	1	-.4811	1

5. THOMAS-GART INTERVALS FOR $N_1 = N_2 = 2$

x_1	x_2	90%		95%		99%	
0	2	-1	.1178	-1	.2515	-1	.4811
0	1	-1	.3486	-1	.4150	-1	.4808
1	2	-1	.3486	-1	.4150	-1	.4808
1	1	-.7151	.7151	-.7945	.7945	-.9043	.9043
0	0	-1	1	-1	1	-1	1
2	2	-1	1	-1	1	-1	1
2	1	-.3486	1	-.4150	1	-.4808	1
1	0	-.3486	1	-.4150	1	-.4808	1
2	0	-.1178	1	-.2515	1	-.4811	1

$$P_{3/4,1/2}\,[\Delta_L(\underset{\sim}{X}) < 1/2 < \Delta_U(\underset{\sim}{X})] = \begin{cases} .996, & \text{based on Table 3} \\ .996, & \text{based on Table 4} \\ .949, & \text{based on Table 5.} \end{cases}$$

Our intervals gain extra coverage probability as α decreases since they become much wider than the TG intervals for $m = 1$ and 3. In general our conditional intervals and the TG intervals are both $(-1,1)$ when $m = 0$ or $N_1 + N_2$; they coincide in a non-trivial interval when $\psi_L < 1 < \psi_U$ and $m = N_1 = N_2$. For other choices of m when $N_1 \geq N_2$ and $N_1 > 1$ the unconditional intervals become much wider than the TG intervals as α decreases; this characteristic is the source of the counterexample given in Section 1. For fixed (x_1,x_2), $\Delta_L(x_1,x_2)$ should intuitively approach -1 and $\Delta_U(x_1,x_2)$ should approach 1 as α decreases to zero. It will be shown below that for a given m, Δ_U and Δ_L generated by the TG method are constrained to a <u>proper</u> subset of $(-1,1)$ unless $m = N_1 = N_2$ and hence cannot attain $+1$ and -1 respectively as α decreases. It follows that counterexamples similar to Example 1.2 are possible even for large N_1 and N_2 by examining Δ near $+1$ and -1, for these values will be excluded from certain Δ intervals regardless of the α chosen.

Our conditional intervals are generally wider than our unconditional intervals although this is not uniformly the case as the outcome $(X_1,X_2) = (1,1)$ shows when $\alpha < .1$. This phenomenon can be explained by looking back at the example. In iteration 2, $S = \{(1,1)\}$ is chosen. If instead $S = \{(0,0),(2,2)\}$ had been used (in violation of (4.3)) the following changes would result in the unconditional intervals:

x_1	x_2	95%		99%	
1	1	-.7623	.7623	-.8946	.8946
0	0	-.8048	.8048	-.9160	.9160
2	2	-.8048	.8048	-.9160	.9160

These unconditional intervals are uniformly more narrow than the conditional intervals of Table 4. However the revised set of unconditional intervals is not uniformly more narrow than the original unconditional intervals of Table 3 (nor is the reverse true). Furthermore the use of the revised unconditional intervals over those of Table 3 results in an increased total (average) length of the intervals from 14.9254 (1.6584) to 14.9522 (1.6614) when $\alpha = .01$ and from 12.5870 (1.3986) to 12.6156 (1.4017) when $\alpha = .05$. This computation illustrates the operation of the "greedy" heuristic in the form of (4.3).

We conclude this section by showing that Δ_U and Δ_L are bounded away from $+1$ and -1 respectively except when $m = N_1 = N_2$. Assume wlog that $N_1 \geq \max\{N_2,2\}$. Fix (x_1,x_2) satisfying $0 < m = x_1 + x_2 < N_1 + N_2$ (the other two cases are trivial). Given $0 < \psi_L \leq \psi_U < \infty$ then the x_L, x_U calculated from (1.5) satisfy:

$$(4.1) \qquad \max\{0,m-N_2\} < x_L, x_U < \min\{m,N\}.$$

Then $\Delta = \Delta_U$ and Δ_L are calculated from $x = x_U$ and x_L respectively by

$$(4.2) \qquad \Delta = \frac{x}{N_1} - \frac{(m-x)}{N_2} = x\left(\frac{1}{N_1} + \frac{1}{N_2}\right) - \frac{m}{N_2}\,.$$

Substituting the bounds (4.1) into the equation (4.2) gives the following bounds on Δ_L and Δ_U:

$$\max\{\frac{-m}{N_2}, \frac{m-N_1-N_2}{N_1}\} < \Delta_L,\Delta_U < \min\{\frac{N_1+N_2-m}{N_2}, \frac{m}{N_1}\}.$$

When $m = N_1 = N_2$ these bounds are -1 and $+1$; when $m \neq N_2$ the lower limit is greater than -1 and when $m \neq N_1$ the upper limit is less than $+1$. Hence if for some $\Delta < 1$ there is a nonempty set $A \subset N_1 \times N_2$ for which $\underset{\sim}{x} \in A \Rightarrow \Delta \notin (\Delta_L(\underset{\sim}{x}),\Delta_U(\underset{\sim}{x}))$ for all α then for any $\varepsilon > 0$, p_1 can be chosen to satisfy

$$P_{p_1,\Delta}[\Delta_L(\underset{\sim}{X}) < \Delta < \Delta_U(\underset{\sim}{X})] \leq \inf_{\pi\in I(\Delta)} P_{\pi,\Delta}[X \notin A] + \varepsilon$$

$$= 1 - \Phi(A,\Delta) + \varepsilon.$$

It follows that for any $\alpha < \Phi(A,\Delta)$, $(\Delta_L(\underset{\sim}{X}), \Delta_U(\underset{\sim}{X}))$ cannot satisfy (1.7).

5. Summary

This paper has adopted a frequentist approach to the problem of determining exact confidence intervals for $\Delta = p_1 - p_2$ in 2×2 contingency tables. Since this is a nuisance parameter problem the intervals proposed achieve converage probabilities greater than or equal to their nominal $(1-\alpha)$ levels. The conditional intervals of Section 2 are easily computed from conditional ψ intervals. The unconditional intervals of Section 3 are much more difficult to compute but generally yield narrower intervals than the conditional ones. The exact method of Thomas and Gart (1977) should be considered an asymptotic method appropriate for reasonably large α. Conditional intervals for ρ are also presented.

Appendix

Two preliminary lemmas are presented below which are required in the proof of Lemma 3.2.

Lemma A.1. Suppose R_i satisfies Condition 3.3; denote $\Pi_1(R_i|x_2) = \{x_1|(x_1,x_2) \in R_i\}$ and $\Pi_2(R_i|x_1) = \{x_2|(x_1,x_2) \in R_i\}$. Whenever these sets are nonempty they are intervals of integers. The proof follows from an easy contradiction argument.

Lemma A.2. Suppose $R = \mathcal{U}' - \mathcal{U}$ where $\mathcal{U} \subset \mathcal{U}' \neq N_1 \times N_2$ and both are NWC sets. For $p_1(\Delta) = \Delta$ then either $G(\Delta) = P_{\Delta,\Delta}[(X_1,X_2) \in R] = 0$ for all $\Delta \in (0,1)$ or there exist integers s,ℓ and $m > 0$ satisfying $0 \leq s \leq \ell \leq m$ for which $G(\Delta) = \sum_{\Delta=s}^{\ell} \binom{m}{j}\Delta^j(1-\Delta)^{m-j}$ for all $\Delta \in (0,1)$. Furthermore analagous representations hold when $p_1(\Delta) = 1$ and $\Delta > 0$, or $p_1 = 0$ and $\Delta < 0$ or $p_1(\Delta) = 1+\Delta$ and $\Delta < 0$.

Proof. By Lemma A.1 $\Pi_1(R|N_2)$ is either empty or has the form $\{s,s+1,\ldots,\ell\}$ for some $0 \leq s \leq \ell \leq N_2$. When $p_1 = \Delta$ then $p_2 = p_1-\Delta = \Delta-\Delta = 0$ and hence $G(\Delta) = 0 \;\; \forall \Delta \in (0,1)$ when $\Pi_1(R|N_2) = \phi$ and $G(\Delta) = \sum_{j=s}^{\ell} \binom{m}{j}\Delta^j(1-\Delta)^{m-j} \;\; \forall \Delta \in (0,1)$ when $\Pi_1(R|N_2) \neq \phi$. This completes the proof.

Proof of Lemma 3.2. First we consider the case when (p_1^*,Δ^*) is on the boundary of $\beta \equiv \{(p_1,\Delta)|\; \Delta \in (-1,1)$ and $p_1 \in \mathcal{I}(\Delta)\}$. Suppose $\Delta^* > 0$ and $p_1^* = \Delta^*$; a slight modification of the argument below works for any $\Delta^* \neq 0$ and $p_1^* \in c\ell(\mathcal{I}(\Delta^*)) - \mathcal{I}(\Delta^*)$. From Lemma A.2, $G(\Delta) = P_{\Delta,\Delta}[(X_1,X_2) \in R]$ is either $0 \;\; \forall \Delta \in (0,1)$ or has the form $\sum_{j=s}^{\ell} \binom{m}{j}\Delta^j(1-\Delta)^{m-j}$ for some m and $0 \leq s \leq \ell \leq m$. Hence if $G(\Delta^*) = 0$ then $G(\Delta) = 0$ for $\Delta \in (0,1)$. We claim $G(\Delta^*) > 0$ is impossible. If $G(\Delta^*) > 0$ then the second

representation holds with $0 \leq s \leq \ell \leq m$. Note that $G'(\Delta^*) = 0$ and $G''(\Delta^*) \geq 0$ since Δ^* is a local minimum. If $s = 0$ and $\ell = m$ then $G(\Delta^*) = 1$ which is impossible. If $s = 0$ and $\ell < m$ then $G(\Delta) = 1 - \int_0^\Delta b(u,\ell+1,m)du$ where $b(u,k,n) = \frac{n!}{(k-1)!(n-k)!} u^{k-1}(1-u)^{n-k}$ while if $0 < s$ and $\ell = m$ then $G(\Delta) = \int_0^\Delta b(u,s,m)du$. In either case $G'(\Delta^*) \neq 0$ and hence is impossible. If $0 < s \leq \ell < m$ then $G(\Delta) = \int_0^1 \{b(u,s,m) - b(u,\ell+1,m)\}du$, $G'(\Delta) = b(\Delta,s,m) - b(\Delta,\ell+1,m)$ and after some algebraic manipulation, $G''(\Delta) = G'(\Delta)\frac{\{s-1+\Delta(1-m)\}}{\Delta(1-\Delta)} - \frac{b(\Delta,\ell+1,m)}{\Delta(1-\Delta)}\{\ell+1-s\}$ $< \frac{G'(\Delta)\{s-1+\Delta(1-m)\}}{\Delta(1-\Delta)}$. In particular $G''(\Delta^*) < 0$ since $G'(\Delta^*) = 0$ which is again impossible. Now suppose $\Delta^* = 0$ and $p_1^* = 0$, then $p_2^* = 0$. Hence $G(0) = \phi(R,0) = 0$ or 1 according as $(0,0) \notin R$ or $(0,0) \in R$. The latter case is impossible since (p_1^*,Δ^*) is a local minimum and $R \neq N_1 \times N_2$. To show that either $\phi(R,\Delta) = 0$ for all $\Delta \in (-1,0)$ or for all $\Delta \in (0,1)$, suppose not. Then there are $-1 < \Delta_1 < 0 < \Delta_2 < 1$ so that $\phi(R,\Delta_i) > 0$ for $i = 1$ and 2. This implies $P_{0,\Delta_1}[(X_1,X_2) \in R] > 0$ and $P_{\Delta_2,\Delta_2}[(X_1,X_2) \in R] > 0$. Hence there exist positive integers x_1^* and x_2^* so that $(0,x_2^*) \in R = \mathcal{U}' - \mathcal{U}$ and $(x_1^*,0) \in R$. Now $(x_1^*,0) \in R$ and $(0,0) \notin R \Rightarrow (0,0) \in \mathcal{U} \Rightarrow (0,x_2) \in \mathcal{U}$ for all $x_2 \in \{0,\ldots,N_2\} \Rightarrow (0,x_2^*) \notin R$ a contradiction. A slight modification of the above argument yields the result when $\Delta^* = 0$ and $p_1^* = 1$.

We now show that a local minimum cannot occur at $(p_1^*,\Delta^*) \in \beta$. Suppose $(p_1^*,\Delta^*) \in \beta$ is a local minimum then $P_{p_1^*,\Delta^*}[(X_1,X_2) \in R] > 0$ since p_1^* is not on boundary of $I(\Delta^*)$. Choose $\varepsilon > 0$ so that the open ball, B, of radius ε satisfies (1) $B \subset \beta$ and (2) $P_{p_1^*,\Delta^*}[(X_1,X_2) \in R] \leq P_{p_1,\Delta}[(X_1,X_2) \in R]$ whenever $(p_1,\Delta) \in B$. Define $p_2^* = p_1^* - \Delta^*$ and for $i \in \Pi_1(R)$ let $s_i = \min\Pi_2(R|i)$ and

$\ell_i = \max \Pi_2(R|i)$; also let $H(p_1,p_2)$ be defined on $[0,1] \times [0,1]$ by

$$H(p_1,p_2) = P_{p_1,p_1-p_2}[(X_1,X_2) \in R]$$

$$= \sum_{i\in C} \binom{N_1}{i} p_1^i (1-p_1)^{N_1-i} \int_{p_2}^{1} b(u,\ell_i+1,N_2)du +$$

$$\sum_{i\in M} \binom{N_1}{i} p_1^i (1-p_1)^{N_1-i} \int_{0}^{p_2} [b(u,s_i,N_2) - b(u,\ell_i+1,N_2)]du +$$

$$\sum_{i\in T} \binom{N_1}{i} p_1^i (1-p_1)^{N_1-i} \int_{0}^{p_2} b(u,s_i,N_2)du +$$

$$\sum_{i\in A} \binom{N_1}{i} p_1^i (1-p_1)^{N_1-i}$$

where $C = \{i \in \Pi_1(R) | 0 = s_i \le \ell_i < N_2\}$, $M = \{i \in \Pi_1(R) | 0 < s_i \le \ell_i < N_2\}$, $T = \{i \in \Pi_1(R) | 0 < s_i \le \ell_i = N_2\}$ and $A = \{i \in \Pi_1(R) | 0 = s_i$ and $\ell_i = N_2\}$.

The case $B = M = T = \phi$ is impossible since $H(p_1,p_2)$ must then be independent of p_2 of the form $H(p_1,p_2) = \sum_{i\in A} \binom{N_1}{i} p_1^i (1-p_1)^{N_1-i} = \sum_{i=s}^{\ell} \binom{N_1}{i} p_1^i (1-p_1)^{N_1-i}$ for some $0 \le s \le \ell \le N_1$ by Condition 3.3. In particular p_1^* must be a local minimum for $G(p_1) = \sum_{i=s}^{\ell} \binom{N_1}{i} p_1^i (1-p_1)^{N_1-i}$. Arguments similar to those above, show this is impossible.

Hence at least one of the set $R_S = \{(i,s_i-1) | i \in \Pi_i(R)$ with $s_i > 0\}$ and $R_L = \{(i,\ell_i) | i \in \Pi_1(R)$ with $\ell_i < N_2\}$ must be nonempty. Now since (p_1^*,Δ^*) is a local minimum it follows that $\nabla H = \nabla H(p_1^*,p_2^*) = 0$ and $\nabla^2 H(p_1^*,p_2^*)$ is positive semi-definite, i.e., $z'\nabla Hz \ge 0$ for all $y \in R^2$. Let $z = (\lambda,-1)$ then

$$z'\nabla Hz = \left[\frac{\partial^2 H}{\partial p_2^2} - \lambda \frac{\partial^2 H}{\partial p_1 \partial p_2}\right] + \lambda^2 \left[\frac{\partial^2 H}{\partial p_1^2} - \frac{1}{\lambda} \frac{\partial^2 H}{\partial p_2 \partial p_1}\right].$$

Let $x_2 = \alpha + \beta i$ $(\beta > 0)$ be the line specified by Condition 3.3 which passes through R and let $\lambda = \beta p_1(1-p_1)/p_2(1-p_2)$. Then

$$\text{(A.1)} \quad z'\nabla^2 Hz = \left[\frac{\partial^2 H}{\partial p_2^2} - \frac{\beta p_1(1-p_1)}{p_2(1-p_2)}\frac{\partial^2 H}{\partial p_1 \partial p_2} - \frac{\alpha}{p_1(1-p_1)}\frac{\partial H}{\partial p_2}\right]$$

$$+ \left[\frac{\beta p_2(1-p_2)}{p_1(1-p_1)}\right]^2 \left[\frac{\partial^2 H}{\partial p_1^2} - \frac{1}{\beta}\frac{p_2(1-p_2)}{p_1(1-p_1)}\frac{\partial^2 H}{\partial p_2 \partial p_1} + \frac{\alpha}{\beta p_1(1-p_1)}\frac{\partial H}{\partial p_2}\right]$$

since $\nabla H = 0$. All derivatives are evaluated at (p_1^*, p_2^*) in (A.1). Both bracketed terms will be shown below to be negative thus leading to the desired contradiction.

After two differentiations and a rearrangement of terms, the first bracketed expression in (A.1) can be shown to be

$$\text{(A.2)} \quad N_2 \sum_{(i,j)\in R_S} \left[\frac{s_i - (\alpha+\beta i)}{p_2^*(1-p_2^*)}\right] \binom{N_1}{i}(p_1^*)^i(1-p_1^*)^{N_1-i}\binom{N_2-1}{s_i-1}(p_2^*)^{s_i-1}(1-p_2^*)^{N_2-s_i}$$

$$- N_2 \sum_{(i,\Delta)\in R_L} \left[\frac{\ell_i + 1 - (\alpha+\beta i)}{p_2^*(1-p_2)^*}\right] \binom{N_1}{i}(p_1^*)^i(1-p_1^*)^{N_1-i}\binom{N_2-1}{\ell_i}(p_2^*)^{\ell_i}(1-p_2^*)^{N_2-1-\ell_i}$$

$$- \left[\frac{1 - N_1 p_1^* + (N_2-1)p_2^*}{p_2^*(1-p_2)^*}\right] \frac{\partial H}{\partial p_2} .$$

By assumption $\frac{\partial H}{\partial p_2} = 0$, $s_i - (\alpha+\beta i) \le 0$ for $(i,j) \in R_S$ and $\ell_i - (\alpha+\beta i) \ge 0$ for $(i,j) \in R_L$. Furthermore $P_{p_1^*,\Delta^*}[(X_1, X_2-1) \in R_L]$ can be shown to be positive from $\frac{\partial H}{\partial p_1} = 0$ and the fact that $R_S \cup R_L$ must be nonempty. Dropping the first term, rewriting the second as a probability and setting the last equal to zero gives the following upper bound on (A.2):

$$\frac{-N_2}{p_2^*(1-p_2^*)} P_{p_1^* p_2^*}[(X_1, X_2 - 1) \in R_L] < 0.$$

A similar argument shows

$$\frac{\partial^2 H}{\partial p_1^2} - \frac{p_2(1-p_2)}{\beta p_1(1-p_1)} \frac{\partial^2 H}{\partial p_2 \partial p_1} + \frac{\alpha}{\beta p_1(1-p_1)} \frac{\partial H}{\partial p_1} < 0$$

and the proof is completed.

The following two lemmas establish that $R_i \cup S$ satisfies Condition 3.3 for $S = L_i - L_0$.

<u>Lemma A.3</u>. Suppose $R_i = \mathcal{U}_i' - \mathcal{U}_i$ satisfies Condition 3.3 and $\phi(R_i, d_{i-1}) \geq 1-\alpha$ where $\alpha \in (0,1)$ and $d_{i-1} \leq 0$; if $(x_1, x_2) \in L_i$ then $(x_1 - 1, x_2) \notin \mathcal{U}_i$ and $(x_1, x_2 + 1) \notin \mathcal{U}_i$ whenever these latter two points are in $N_1 \times N_2$.

<u>Proof</u>. Suppose R_i satisfies the above conditions and there exists $(x_1, x_2) \in L_i$ which $(x_1 - 1, x_2) \in \mathcal{U}_i$ (the case $(x_1, x_2 + 1)$ is proved analogously). By assumption $\Pi_2(R_i) = \{j \text{ integer} | s \leq j \leq \ell\}$ for some $0 \leq s \leq \ell \leq N_2$. For any $k = 0, \ldots, x_1 - 1$ we have $(k, x_2) \in \mathcal{U}_i \Rightarrow (k, x_2) \notin R_i$ $\Rightarrow$ either $x_2 < s$ or $x_2 > \ell$. If $x_2 < s \Rightarrow (0, \ell) \notin R_i$ for $\ell \leq x_2$; $(x_1 - 1, x_2) \in \mathcal{U}_i \Rightarrow (0, x_2) \in \mathcal{U}_i \Rightarrow (0, \ell) \notin R_i$ for $\ell > x_2$ and we conclude that the entire line $\{(0, \ell) | \ell = 0, \ldots, N_2\}$ is not in R_i. But this implies

$$0 < 1-\alpha \leq \phi(R_i, d_{i-1}) \leq P_{0, d_{i-1}}[(X_1, X_2) \in R_i] = 0$$

a contradiction. If $x_2 > \ell$ then a similar contradiction results and the proof if completed.

<u>Lemma A.4</u>. If R_i satisfies the conditions of Lemma A.3 then $R_i \cup S$ satisfies Condition 3.3 for $S = L_i - L_0$.

<u>Proof</u>. To show part (1) of Condition 3.3 it suffices to prove that $N_1 \times N_2 - L_0$ is a NWC set since $R_i \cup (L_i - L_0) = (N_1 \times N_2 - L_0) - \mathcal{U}_i$ and $\mathcal{U}_i \subset \mathcal{U}_0 \subset N_1 \times N_2 - L_0$. Pick $(x_1,x_2) \in N_1 \times N_2 - L_0$ and any integers $\ell(1 \le \ell \le N_2 - x_2)$ and $k(1 \le k \le x_1)$; it must be shown that $(x_1, x_2+\ell)$ and $(x_1-k, x_2) \in N_1 \times N_2 - L_0$. We have $(x_1,x_2) \in N_1 \times N_2 - L_0 \Longleftrightarrow (x_1,x_2) \notin L_0 = \mathcal{U}_0^\rho \Longleftrightarrow (N_1-x_1, N_2-x_2) \notin \mathcal{U}_0$ $\Rightarrow (N_1-x_1+k, N_2-x_2) \notin \mathcal{U}_0$ and $(N_1-x_1, N_2-x_2-\ell) \notin \mathcal{U}_0 \Longleftrightarrow (x_1-k, \ell_2) \notin \mathcal{U}_0^\rho = L_0$ and $(x_1, x_2+\ell) \notin L_0 \Longleftrightarrow (x_1-k, x_2)$ and $(x_1, x_2+\ell) \in N_1 \times N_2 - L_0$. We next show that $\Pi_1(R_i \cup S)$ must be an interval of integers. Let $\underline{x}_1 = \min \Pi_1(R_i \cup S)$ and $\overline{x}_1 = \max \Pi_1(R_i \cup S)$. Suppose there exists an integer $\hat{x}_1$, $\underline{x}_1 < \hat{x}_1 < \overline{x}_1$, for which $\hat{x}_1 \notin \Pi_1(R_i \cup S)$. It follows that each $(\hat{x}_1, \ell)$ must be in L_0 or $\mathcal{U}_i$ for $\ell = 0,\ldots,N_2$. We claim that $(\hat{x}_1, N_2) \notin L_0$ and $(\hat{x}_1, 0) \notin \mathcal{U}_i$. If $(\hat{x}_1, N_2) \in L_0$ then $(\hat{x}_1, N_2) \notin N_1 \times N_2 - L_0 \Rightarrow (\overline{x}_1, \ell) \notin N_1 \times N_2 - L_0 \subset R_i \cup S$ for every ℓ contradicting the assumption that $\overline{x}_1 \in \Pi_1(R_i \cup S)$. Similarly if $(\hat{x}_1, 0) \in \mathcal{U}_i$ then $(i,j) \in \mathcal{U}_i$ for all integers $0 \le i \le \hat{x}_1$ and $0 \le j \le N_2$ contradicting the assumption $\underline{x}_1 \in \Pi_1(R_i \cup S)$. Hence it must be that $(\hat{x}_1, 0) \in L_0$ and $(\hat{x}_1, N_2) \in \mathcal{U}_i$. It follows that $\hat{x}_2 \equiv \min \Pi_1(\mathcal{U}_i | \hat{x}_1)$ is (strictly) positive and $(\hat{x}_1, \hat{x}_2 - 1) \in L_0 \subset L_i$. But this contradicts Lemma A.3 and $(\hat{x}_1, \hat{x}_2) \in \mathcal{U}_i$. A similar argument shows that $\Pi_2(R_i \cup S)$ is an interval of integers and concludes the proof that part (2) of Condition 3 holds. We begin the proof that part (3) of Condition 3.3 holds by choosing $\alpha \in R^1$ and $\beta > 0$ so that part (3) holds for R_i. Let s_k, ℓ_k, S_j and L_j be defined for R_i as in part (3) and s_k^*, ℓ_k^*, S_j^* and L_j^* be defined for $R_i \cup S$ in a similar fashion. Pick $m \in \Pi_1(R_i \cup S)$ having $0 < s_m^*$. If $m \in \Pi_1(R_i)$ then $s_m^* \le s_m$ since $R_i \subset S \cup R_i \Rightarrow s_m > 0$ and so $s_m^* \le s_m \le \alpha + \beta m$. Now suppose $m \notin \Pi_1(R_i)$ then $(m, s_m^*) \in L_i - L_0$. Two subcases must be considered: (1) $s_m^* \in \Pi_2(R_i)$ and (2) $s_m^* \notin \Pi_2(R_i)$. In the first subcase we $L_{s_m^*} < m$ or $m < S_{s_m^*}$ since $(m, s_m^*) \notin R_i$. The case $m < S_{s_m^*}$ is impossible since

$(S_{s^*_m}, s^*_m) \in \mathcal{U}'_i \Rightarrow (m, s^*_m) \in \mathcal{U}'$ contradicting the assumption $(m, s^*_m) \in L_i$. So $L_{s^*_m} < m \leq N_2$ and since R_i satisfies Condition 3.3 it follows that $L^*_{s^*_m} \geq m > L_{s^*_m} \geq \frac{s^*_m - \alpha}{\beta}$ or $\alpha + \beta m > s^*_m$. In subcase 2 it must be that $\{(0, s^*_m), \ldots, (m, s^*_m)\} \subset L_i - L_0$ otherwise there exists an x^*_1 satisfying $(x^*_1, s^*_m) \in \mathcal{U}_i$ and $(x^*_1 + 1, s^*_m) \in L_i$ a contradiction. This shows that $R_i \subset \{(y_1, y_2) | 0 \leq y_1 < m \text{ and } s^*_m < y_2 \leq N_2\}$. So for any $y_1 \in \Pi_1(R_i)$ we have $y_1 < m$ and $s_{y_1} > s^*_m \geq 0$ and hence $s^*_m < s_{y_1} \leq \alpha + \beta y_1 < \alpha + \beta m \Rightarrow s^*_m < \alpha + \beta m$. The remaining three cases follow from analogous arguments and complete the proof.

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1. CONFIDENCE INTERVALS FOR THE DIFFERENCE OF TWO PROPORTIONS

N1	N2	X1	X2	90%		95%		99%	
1	1	1	0	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
2	1	2	0	-0.3592	1.0000	-0.5101	1.0000	-0.7511	1.0000
2	1	1	1	-1.0000	0.6185	-1.0000	0.7208	-1.0000	0.8673
2	1	1	0	-0.6185	1.0000	-0.7208	1.0000	-0.8673	1.0000
2	2	2	1	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
2	2	2	0	-0.1178	1.0000	-0.2515	1.0000	-0.4811	1.0000
2	2	1	2	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
2	2	1	1	-0.7151	0.7151	-0.7945	0.7945	-0.9043	0.9043
2	2	1	0	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
3	1	3	0	-0.2679	1.0000	-0.4313	1.0000	-0.7035	1.0000
3	1	2	1	-1.0000	0.6772	-1.0000	0.7661	-1.0000	0.8903
3	1	2	0	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
3	2	3	1	-0.4202	1.0000	-0.5613	1.0000	-0.7808	1.0000
3	2	3	0	-0.0000	1.0000	-0.1330	1.0000	-0.3746	1.0000
3	2	2	2	-1.0000	0.5721	-1.0000	0.6845	-1.0000	0.8483
3	2	2	1	-0.6199	0.7614	-0.7214	0.8291	-0.8673	0.9212
3	2	2	0	-0.2679	1.0000	-0.3872	1.0000	-0.5839	1.0000
3	3	3	2	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
3	3	3	1	-0.1531	1.0000	-0.2763	1.0000	-0.4906	1.0000
3	3	3	0	0.0	1.0000	-0.0000	1.0000	-0.2310	1.0000
3	3	2	3	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
3	3	2	2	-0.6789	0.6789	-0.7667	0.7667	-0.8904	0.8904
3	3	2	1	-0.4074	0.8011	-0.5082	0.8583	-0.6708	0.9352
3	3	2	0	-0.1531	1.0000	-0.2763	1.0000	-0.4906	1.0000
4	1	4	0	-0.2000	1.0000	-0.3710	1.0000	-0.6653	1.0000
4	1	3	1	-1.0000	0.7143	-1.0000	0.7942	-1.0000	0.9043
4	1	3	0	-0.4202	1.0000	-0.5613	1.0000	-0.7808	1.0000
4	1	2	1	-1.0000	0.5721	-1.0000	0.6845	-1.0000	0.8483
4	1	2	0	-0.5721	1.0000	-0.6845	1.0000	-0.8483	1.0000
4	2	4	1	-0.3592	1.0000	-0.5101	1.0000	-0.7511	1.0000
4	2	4	0	0.0	1.0000	-0.0524	1.0000	-0.3000	1.0000

N1	N2	X1	X2	90%		95%		99%	
4	2	3	2	-1.0000	0.6185	-1.0000	0.7208	-1.0000	0.8673
4	2	3	1	-0.5538	0.7901	-0.6692	0.8503	-0.8399	0.9314
4	2	3	0	-0.1531	1.0000	-0.2765	1.0000	-0.4906	1.0000
4	2	2	2	-1.0000	0.3527	-1.0000	0.4625	-1.0000	0.6396
4	2	2	1	-0.6789	0.6789	-0.7667	0.7667	-0.8904	0.8904
4	2	2	0	-0.3527	1.0000	-0.4625	1.0000	-0.6396	1.0000
4	3	4	2	-0.4441	1.0000	-0.5811	1.0000	-0.7920	1.0000
4	3	4	1	-0.0740	1.0000	-0.1990	1.0000	-0.4238	1.0000
4	3	4	0	0.0	1.0000	0.0	1.0000	-0.1408	1.0000
4	3	3	3	-1.0000	0.5520	-1.0000	0.6685	-1.0000	0.8399
4	3	3	2	-0.6206	0.7159	-0.7216	0.7948	-0.8674	0.9044
4	3	3	1	-0.3013	0.8255	-0.4089	0.8762	-0.5915	0.9436
4	3	3	0	-0.1150	1.0000	-0.1437	1.0000	-0.3542	1.0000
4	3	2	3	-1.0000	0.2425	-1.0000	0.3585	-1.0000	0.5550
4	3	2	2	-0.7303	0.4829	-0.8055	0.5732	-0.9096	0.7168
4	3	2	1	-0.4829	0.7303	-0.5732	0.8055	-0.7168	0.9096
4	3	2	0	-0.2425	1.0000	-0.3585	1.0000	-0.5550	1.0000
4	4	4	3	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
4	4	4	2	-0.1654	1.0000	-0.2846	1.0000	-0.4937	1.0000
4	4	4	1	0.0	1.0000	-0.0571	1.0000	-0.2679	1.0000
4	4	4	0	0.0	1.0000	0.0	1.0000	-0.0435	1.0000
4	4	3	4	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
4	4	3	3	-0.6631	0.6631	-0.7544	0.7544	-0.8841	0.8841
4	4	3	2	-0.3840	0.7623	-0.4822	0.8294	-0.6462	0.9212
4	4	3	1	-0.2270	0.8472	-0.2821	0.8919	-0.4654	0.9510
4	4	3	0	0.0	1.0000	-0.0571	1.0000	-0.2679	1.0000
4	4	2	4	-1.0000	0.1654	-1.0000	0.2846	-1.0000	0.4937
4	4	2	3	-0.7623	0.3840	-0.8294	0.4822	-0.9212	0.6462
4	4	2	2	-0.5520	0.5520	-0.6317	0.6317	-0.7574	0.7574
4	4	2	1	-0.3840	0.7623	-0.4822	0.8294	-0.6462	0.9212
4	4	2	0	-0.1654	1.0000	-0.2846	1.0000	-0.4937	1.0000
5	1	5	0	-0.1459	1.0000	-0.3219	1.0000	-0.6330	1.0000
5	1	4	1	-1.0000	0.7405	-1.0000	0.8139	-1.0000	0.9140
5	1	4	0	-0.3592	1.0000	-0.5101	1.0000	-0.7511	1.0000
5	1	3	1	-1.0000	0.6185	-1.0000	0.7208	-1.0000	0.8673
5	1	3	0	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
5	2	5	1	-0.3097	1.0000	-0.4676	1.0000	-0.7258	1.0000
5	2	5	0	0.0	1.0000	0.0	1.0000	-0.2421	1.0000

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N1	N2	X1	X2	90%		95%		99%	
5	2	4	2	-1.0000	0.6518	-1.0000	0.7466	-1.0000	0.8805
5	2	4	1	-0.5020	0.8102	-0.6276	0.8650	-0.8174	0.9384
5	2	4	0	-0.0740	1.0000	-0.1990	1.0000	-0.4238	1.0000
5	2	3	2	-1.0000	0.4105	-1.0000	0.5132	-1.0000	0.6765
5	2	3	1	-0.6206	0.7159	-0.7216	0.7948	-0.8674	0.9044
5	2	3	0	-0.2425	1.0000	-0.3583	1.0000	-0.5550	1.0000
5	3	5	2	-0.3983	1.0000	-0.5430	1.0000	-0.7703	1.0000
5	3	5	1	-0.0137	1.0000	-0.1393	1.0000	-0.3710	1.0000
5	3	5	0	0.0	1.0000	0.0	1.0000	-0.0742	1.0000
5	3	4	3	-1.0000	0.5896	-1.0000	0.6982	-1.0000	0.8555
5	3	4	2	-0.5746	0.7422	-0.6854	0.8145	-0.8484	0.9140
5	3	4	1	-0.2264	0.8426	-0.3381	0.8885	-0.5337	0.9494
5	3	4	0	0.0	1.0000	-0.0571	1.0000	-0.2679	1.0000
5	3	3	3	-1.0000	0.3047	-1.0000	0.4147	-1.0000	0.5984
5	3	3	2	-0.6797	0.5333	-0.7670	0.6162	-0.8904	0.7470
5	3	3	1	-0.3840	0.7623	-0.4822	0.8294	-0.6462	0.9212
5	3	3	0	-0.1896	1.0000	-0.2294	1.0000	-0.4258	1.0000
5	4	5	3	-0.4570	1.0000	-0.5917	1.0000	-0.7980	1.0000
5	4	5	2	-0.1059	1.0000	-0.2272	1.0000	-0.4448	1.0000
5	4	5	1	0.0	1.0000	0.0	1.0000	-0.2035	1.0000
5	4	5	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
5	4	4	4	-1.0000	0.5407	-1.0000	0.6595	-1.0000	0.8350
5	4	4	3	-0.6210	0.6933	-0.7218	0.7775	-0.8674	0.8957
5	4	4	2	-0.3132	0.7848	-0.4165	0.8461	-0.5941	0.9293
5	4	4	1	-0.1150	0.8623	-0.1980	0.9028	-0.3857	0.9561
5	4	4	0	0.0	1.0000	0.0	1.0000	-0.1714	1.0000
5	4	3	4	-1.0000	0.2298	-1.0000	0.3443	-1.0000	0.5412
5	4	3	3	-0.7168	0.4401	-0.7951	0.5316	-0.9044	0.6825
5	4	3	2	-0.4612	0.5975	-0.5494	0.6701	-0.6950	0.7838
5	4	3	1	-0.2981	0.7911	-0.3616	0.8506	-0.5286	0.9314
5	4	3	0	-0.0762	1.0000	-0.1440	1.0000	-0.3438	1.0000
5	5	5	4	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
5	5	5	3	-0.1716	1.0000	-0.2888	1.0000	-0.4953	1.0000
5	5	5	2	0.0	1.0000	-0.0800	1.0000	-0.2818	1.0000
5	5	5	1	0.0	1.0000	0.0	1.0000	-0.1012	1.0000
5	5	5	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
5	5	4	5	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
5	5	4	4	-0.6543	0.6543	-0.7475	0.7475	-0.8806	0.8806
5	5	4	3	-0.3727	0.7430	-0.4698	0.8148	-0.6345	0.9141
5	5	4	2	-0.1896	0.8111	-0.2808	0.8654	-0.4542	0.9385
5	5	4	1	-0.0124	0.8760	-0.1648	0.9126	-0.2926	0.9606
5	5	4	0	0.0	1.0000	0.0	1.0000	-0.1012	1.0000

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N1	N2	X1	X2	90%		95%		99%	
5	5	3	5	-1.0000	0.1716	-1.0000	0.2888	-1.0000	0.4953
5	5	3	4	-0.7430	0.3727	-0.8148	0.4698	-0.9141	0.6345
5	5	3	3	-0.5128	0.5128	-0.5942	0.5942	-0.7271	0.7271
5	5	3	2	-0.3660	0.6394	-0.4361	0.7053	-0.5864	0.8077
5	5	3	1	-0.1896	0.8111	-0.2808	0.8654	-0.4542	0.9385
5	5	3	0	0.0	1.0000	-0.0800	1.0000	-0.2818	1.0000
6	1	6	0	-0.1010	1.0000	-0.2804	1.0000	-0.6049	1.0000
6	1	5	1	-1.0000	0.7604	-1.0000	0.8287	-1.0000	0.9212
6	1	5	0	-0.3097	1.0000	-0.4676	1.0000	-0.7258	1.0000
6	1	4	1	-1.0000	0.6518	-1.0000	0.7466	-1.0000	0.8805
6	1	4	0	-0.4441	1.0000	-0.5811	1.0000	-0.7920	1.0000
6	1	3	1	-1.0000	0.5520	-1.0000	0.6685	-1.0000	0.8399
6	1	3	0	-0.5520	1.0000	-0.6685	1.0000	-0.8399	1.0000
6	2	6	1	-0.2679	1.0000	-0.4313	1.0000	-0.7035	1.0000
6	2	6	0	0.0	1.0000	0.0	1.0000	-0.1946	1.0000
6	2	5	2	-1.0000	0.6772	-1.0000	0.7661	-1.0000	0.8903
6	2	5	1	-0.4593	0.8254	-0.5926	0.8761	-0.7981	0.9436
6	2	5	0	-0.0137	1.0000	-0.1393	1.0000	-0.3710	1.0000
6	2	4	2	-1.0000	0.4536	-1.0000	0.5508	-1.0000	0.7035
6	2	4	1	-0.5746	0.7422	-0.6854	0.8145	-0.8484	0.9140
6	2	4	0	-0.1654	1.0000	-0.2846	1.0000	-0.4937	1.0000
6	2	3	2	-1.0000	0.3047	-1.0000	0.4147	-1.0000	0.5984
6	2	3	1	-0.6631	0.6631	-0.7544	0.7544	-0.8841	0.8841
6	2	3	0	-0.3047	1.0000	-0.4147	1.0000	-0.5984	1.0000
6	3	6	2	-0.3592	1.0000	-0.5101	1.0000	-0.7511	1.0000
6	3	6	1	0.0	1.0000	-0.0908	1.0000	-0.3271	1.0000
6	3	6	0	0.0	1.0000	0.0	1.0000	-0.0216	1.0000
6	3	5	3	-1.0000	0.6185	-1.0000	0.7208	-1.0000	0.8673
6	3	5	2	-0.5361	0.7620	-0.6547	0.8293	-0.8320	0.9212
6	3	5	1	-0.1682	0.8553	-0.2825	0.8978	-0.4872	0.9538
6	3	5	0	0.0	1.0000	0.0	1.0000	-0.2035	1.0000
6	3	4	3	-1.0000	0.3518	-1.0000	0.4570	-1.0000	0.6303
6	3	4	2	-0.6392	0.5702	-0.7358	0.6477	-0.8745	0.7689
6	3	4	1	-0.3132	0.7848	-0.4165	0.8461	-0.5941	0.9293
6	3	4	0	-0.0762	1.0000	-0.1440	1.0000	-0.3438	1.0000
6	3	3	3	-1.0000	0.2446	-1.0000	0.2898	-1.0000	0.4755
6	3	3	2	-0.7168	0.4401	-0.7951	0.5316	-0.9044	0.6825
6	3	3	1	-0.4401	0.7168	-0.5316	0.7951	-0.6825	0.9044
6	3	3	0	-0.2446	1.0000	-0.2898	1.0000	-0.4755	1.0000

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N1	N2	X1	X2	90%		95%		99%	
6	4	6	3	-0.4202	1.0000	-0.5613	1.0000	-0.7808	1.0000
6	4	6	2	-0.0575	1.0000	-0.1800	1.0000	-0.4038	1.0000
6	4	6	1	0.0	1.0000	0.0	1.0000	-0.1519	1.0000
6	4	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
6	4	5	4	-1.0000	0.5721	-1.0000	0.6845	-1.0000	0.8483
6	4	5	3	-0.5857	0.7164	-0.6940	0.7949	-0.8529	0.9044
6	4	5	2	-0.2575	0.8018	-0.3643	0.8586	-0.5519	0.9352
6	4	5	1	-0.0391	0.8736	-0.1345	0.9109	-0.3254	0.9598
6	4	5	0	0.0	1.0000	0.0	1.0000	-0.1012	1.0000
6	4	4	4	-1.0000	0.2791	-1.0000	0.3893	-1.0000	0.5765
6	4	4	3	-0.6801	0.4818	-0.7671	0.5680	-0.8904	0.7090
6	4	4	2	-0.3953	0.6306	-0.4892	0.6980	-0.6485	0.8029
6	4	4	1	-0.1896	0.8111	-0.2808	0.8654	-0.4542	0.9385
6	4	4	0	0.0	1.0000	-0.0864	1.0000	-0.2496	1.0000
6	4	3	4	-1.0000	0.1333	-1.0000	0.2063	-1.0000	0.3971
6	4	3	3	-0.7502	0.3498	-0.8202	0.4165	-0.9167	0.5719
6	4	3	2	-0.5128	0.5128	-0.5942	0.5942	-0.7271	0.7271
6	4	3	1	-0.3498	0.7502	-0.4165	0.8202	-0.5719	0.9167
6	4	3	0	-0.1333	1.0000	-0.2063	1.0000	-0.3971	1.0000
6	5	6	4	-0.4650	1.0000	-0.5983	1.0000	-0.8016	1.0000
6	5	6	3	-0.1239	1.0000	-0.2431	1.0000	-0.4566	1.0000
6	5	6	2	0.0	1.0000	-0.0288	1.0000	-0.2318	1.0000
6	5	6	1	0.0	1.0000	0.0	1.0000	-0.0462	1.0000
6	5	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
6	5	5	5	-1.0000	0.5334	-1.0000	0.6537	-1.0000	0.8319
6	5	5	4	-0.6213	0.6797	-0.7219	0.7670	-0.8674	0.8904
6	5	5	3	-0.3193	0.7628	-0.4204	0.8296	-0.5954	0.9213
6	5	5	2	-0.1197	0.8262	-0.2193	0.8764	-0.3974	0.9437
6	5	5	1	0.0	0.8862	-0.1003	0.9199	-0.2244	0.9640
6	5	5	0	0.0	1.0000	0.0	1.0000	-0.0893	1.0000
6	5	4	5	-1.0000	0.2221	-1.0000	0.3358	-1.0000	0.5330
6	5	4	4	-0.7092	0.4174	-0.7892	0.5095	-0.9014	0.6642
6	5	4	3	-0.4508	0.5509	-0.5381	0.6270	-0.6846	0.7504
6	5	4	2	-0.2683	0.6698	-0.3597	0.7306	-0.5176	0.8248
6	5	4	1	-0.1049	0.8295	-0.2635	0.8788	-0.3651	0.9448
6	5	4	0	0.0	1.0000	-0.0686	1.0000	-0.1807	1.0000
6	5	3	5	-1.0000	0.0577	-1.0000	0.1432	-1.0000	0.3375
6	5	3	4	-0.7738	0.2447	-0.8377	0.3389	-0.9251	0.5018
6	5	3	3	-0.5611	0.4149	-0.6355	0.4870	-0.7563	0.6256
6	5	3	2	-0.4149	0.5611	-0.4870	0.6355	-0.6256	0.7563
6	5	3	1	-0.2447	0.7738	-0.3389	0.8377	-0.5018	0.9251
6	5	3	0	-0.0577	1.0000	-0.1432	1.0000	-0.3375	1.0000
6	6	6	5	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000

N1	N2	X1	X2	90%		95%		99%	
6	6	6	4	-0.1753	1.0000	-0.2914	1.0000	-0.4962	1.0000
6	6	6	3	0.0	1.0000	-0.0924	1.0000	-0.2891	1.0000
6	6	6	2	0.0	1.0000	0.0	1.0000	-0.1262	1.0000
6	6	6	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
6	6	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
6	6	5	6	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
6	6	5	5	-0.6486	0.6486	-0.7430	0.7430	-0.8783	0.8783
6	6	5	4	-0.3660	0.7312	-0.4624	0.8058	-0.6276	0.9096
6	6	5	3	-0.1838	0.7915	-0.2793	0.8508	-0.4478	0.9314
6	6	5	2	-0.0493	0.8433	-0.1677	0.8888	-0.2994	0.9495
6	6	5	1	0.0	0.8957	0.0	0.9267	-0.1773	0.9671
6	6	5	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
6	6	4	6	-1.0000	0.1753	-1.0000	0.2914	-1.0000	0.4962
6	6	4	5	-0.7312	0.3660	-0.8058	0.4624	-0.9096	0.6276
6	6	4	4	-0.4920	0.4920	-0.5742	0.5742	-0.7109	0.7109
6	6	4	3	-0.3284	0.5964	-0.4143	0.6657	-0.5612	0.7775
6	6	4	2	-0.1968	0.6981	-0.3261	0.7542	-0.4332	0.8406
6	6	4	1	-0.0493	0.8433	-0.1677	0.8888	-0.2994	0.9495
6	6	4	0	0.0	1.0000	0.0	1.0000	-0.1262	1.0000
6	6	3	6	-1.0000	0.0	-1.0000	0.0924	-1.0000	0.2891
6	6	3	5	-0.7915	0.1838	-0.8508	0.2793	-0.9315	0.4478
6	6	3	4	-0.5964	0.3284	-0.6657	0.4143	-0.7775	0.5612
6	6	3	3	-0.4616	0.4616	-0.5347	0.5347	-0.6619	0.6619
6	6	3	2	-0.3284	0.5964	-0.4143	0.6657	-0.5612	0.7775
6	6	3	1	-0.1838	0.7915	-0.2793	0.8508	-0.4478	0.9314
6	6	3	0	0.0	1.0000	-0.0924	1.0000	-0.2891	1.0000
7	1	7	0	-0.0627	1.0000	-0.2446	1.0000	-0.5799	1.0000
7	1	6	1	-1.0000	0.7762	-1.0000	0.8404	-1.0000	0.9268
7	1	6	0	-0.2679	1.0000	-0.4313	1.0000	-0.7035	1.0000
7	1	5	1	-1.0000	0.6772	-1.0000	0.7661	-1.0000	0.8903
7	1	5	0	-0.3983	1.0000	-0.5430	1.0000	-0.7703	1.0000
7	1	4	1	-1.0000	0.5896	-1.0000	0.6982	-1.0000	0.8555
7	1	4	0	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
7	2	7	1	-0.2318	1.0000	-0.3994	1.0000	-0.6835	1.0000
7	2	7	0	0.0	1.0000	0.0	1.0000	-0.1544	1.0000
7	2	6	2	-1.0000	0.6975	-1.0000	0.7815	-1.0000	0.8980
7	2	6	1	-0.4226	0.8373	-0.5623	0.8848	-0.7809	0.9477
7	2	6	0	0.0	1.0000	-0.0908	1.0000	-0.3271	1.0000
7	2	5	2	-1.0000	0.4876	-1.0000	0.5802	-1.0000	0.7243
7	2	5	1	-0.5361	0.7620	-0.6547	0.8293	-0.8320	0.9212
7	2	5	0	-0.1059	1.0000	-0.2272	1.0000	-0.4448	1.0000

N1	N2	X1	X2	90%		95%		99%	
7	2	4	2	-1.0000	0.3518	-1.0000	0.4570	-1.0000	0.6303
7	2	4	1	-0.6210	0.6933	-0.7218	0.7775	-0.8674	0.8957
7	2	4	0	-0.2298	1.0000	-0.3443	1.0000	-0.5412	1.0000
7	3	7	2	-0.3252	1.0000	-0.4810	1.0000	-0.7338	1.0000
7	3	7	1	0.0	1.0000	-0.0500	1.0000	-0.2895	1.0000
7	3	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	3	6	3	-1.0000	0.6417	-1.0000	0.7388	-1.0000	0.8765
7	3	6	2	-0.5029	0.7778	-0.6279	0.8410	-0.8175	0.9269
7	3	6	1	-0.1206	0.8654	-0.2366	0.9050	-0.4481	0.9571
7	3	6	0	0.0	1.0000	0.0	1.0000	-0.1519	1.0000
7	3	5	3	-1.0000	0.3893	-1.0000	0.4904	-1.0000	0.6553
7	3	5	2	-0.6052	0.5990	-0.7091	0.6721	-0.8607	0.7858
7	3	5	1	-0.2575	0.8018	-0.3643	0.8586	-0.5519	0.9352
7	3	5	0	0.0	1.0000	-0.0800	1.0000	-0.2818	1.0000
7	3	4	3	-1.0000	0.2879	-1.0000	0.3359	-1.0000	0.5130
7	3	4	2	-0.6801	0.4818	-0.7671	0.5680	-0.8904	0.7090
7	3	4	1	-0.3727	0.7430	-0.4698	0.8148	-0.6345	0.9141
7	3	4	0	-0.1333	1.0000	-0.2063	1.0000	-0.3971	1.0000
7	4	7	3	-0.3880	1.0000	-0.5343	1.0000	-0.7653	1.0000
7	4	7	2	-0.0169	1.0000	-0.1399	1.0000	-0.3684	1.0000
7	4	7	1	0.0	1.0000	0.0	1.0000	-0.1088	1.0000
7	4	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	4	6	4	-1.0000	0.5975	-1.0000	0.7044	-1.0000	0.8588
7	4	6	3	-0.5550	0.7347	-0.6697	0.8087	-0.8400	0.9112
7	4	6	2	-0.2115	0.8152	-0.3208	0.8684	-0.5161	0.9399
7	4	6	1	0.0	0.8824	-0.0836	0.9172	-0.2765	0.9628
7	4	6	0	0.0	1.0000	0.0	1.0000	-0.0462	1.0000
7	4	5	4	-1.0000	0.3186	-1.0000	0.4252	-1.0000	0.6042
7	4	5	3	-0.6491	0.5146	-0.7432	0.5965	-0.8783	0.7295
7	4	5	2	-0.3429	0.6562	-0.4409	0.7195	-0.6106	0.8175
7	4	5	1	-0.1197	0.8262	-0.2193	0.8764	-0.3974	0.9437
7	4	5	0	0.0	1.0000	-0.0686	1.0000	-0.1807	1.0000
7	4	4	4	-1.0000	0.1788	-1.0000	0.2544	-1.0000	0.4378
7	4	4	3	-0.7172	0.3900	-0.7953	0.4578	-0.9044	0.6043
7	4	4	2	-0.4508	0.5509	-0.5381	0.6270	-0.6846	0.7504
7	4	4	1	-0.2447	0.7738	-0.3389	0.8377	-0.5018	0.9251
7	4	4	0	-0.0186	1.0000	-0.1590	1.0000	-0.3057	1.0000
7	5	7	4	-0.4343	1.0000	-0.5730	1.0000	-0.7874	1.0000
7	5	7	3	-0.0836	1.0000	-0.2040	1.0000	-0.4230	1.0000
7	5	7	2	0.0	1.0000	0.0	1.0000	-0.1898	1.0000
7	5	7	1	0.0	1.0000	0.0	1.0000	-0.0011	1.0000
7	5	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000

N1	N2	X1	X2	90%		95%		99%	
7	5	6	5	-1.0000	0.5604	-1.0000	0.6752	-1.0000	0.8434
7	5	6	4	-0.5906	0.6999	-0.6994	0.7824	-0.8557	0.8981
7	5	6	3	-0.2749	0.7785	-0.3790	0.8412	-0.5622	0.9269
7	5	6	2	-0.0696	0.8381	-0.1695	0.8851	-0.3510	0.9477
7	5	6	1	0.0	0.8943	-0.0319	0.9257	-0.1704	0.9666
7	5	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	5	5	5	-1.0000	0.2630	-1.0000	0.3734	-1.0000	0.5628
7	5	5	4	-0.6804	0.4527	-0.7673	0.5407	-0.8904	0.6873
7	5	5	3	-0.4011	0.5806	-0.4929	0.6525	-0.6497	0.7684
7	5	5	2	-0.2081	0.6933	-0.3009	0.7502	-0.4647	0.8380
7	5	5	1	-0.0493	0.8433	-0.1677	0.8888	-0.2994	0.9495
7	5	5	0	0.0	1.0000	0.0	1.0000	-0.1511	1.0000
7	5	4	5	-1.0000	0.1040	-1.0000	0.1924	-1.0000	0.3803
7	5	4	4	-0.7434	0.2927	-0.8149	0.3830	-0.9141	0.5376
7	5	4	3	-0.5031	0.4528	-0.5837	0.5250	-0.7176	0.6547
7	5	4	2	-0.3284	0.5964	-0.4143	0.6657	-0.5612	0.7775
7	5	4	1	-0.1710	0.7954	-0.3120	0.8536	-0.4163	0.9328
7	5	4	0	0.0	1.0000	-0.1210	1.0000	-0.2383	1.0000
7	6	7	5	-0.4705	1.0000	-0.6028	1.0000	-0.8041	1.0000
7	6	7	4	-0.1355	1.0000	-0.2533	1.0000	-0.4642	1.0000
7	6	7	3	0.0	1.0000	-0.0500	1.0000	-0.2482	1.0000
7	6	7	2	0.0	1.0000	0.0	1.0000	-0.0813	1.0000
7	6	7	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	6	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	6	6	6	-1.0000	0.5283	-1.0000	0.6496	-1.0000	0.8297
7	6	6	5	-0.6215	0.6704	-0.7220	0.7598	-0.8674	0.8868
7	6	6	4	-0.3231	0.7487	-0.4228	0.8190	-0.5963	0.9161
7	6	6	3	-0.1345	0.8055	-0.2308	0.8611	-0.4036	0.9364
7	6	6	2	-0.0058	0.8541	-0.1003	0.8966	-0.2469	0.9531
7	6	6	1	0.0	0.9031	0.0	0.9319	-0.0948	0.9695
7	6	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	6	5	6	-1.0000	0.2170	-1.0000	0.3302	-1.0000	0.5276
7	6	5	5	-0.7043	0.4031	-0.7854	0.4956	-0.8995	0.6527
7	6	5	4	-0.4447	0.5244	-0.5314	0.6024	-0.6784	0.7313
7	6	5	3	-0.2701	0.6238	-0.3580	0.6891	-0.5114	0.7938
7	6	5	2	-0.1466	0.7199	-0.2335	0.7722	-0.3705	0.8526
7	6	5	1	-0.0080	0.8560	-0.0686	0.8980	-0.2372	0.9538
7	6	5	0	0.0	1.0000	0.0	1.0000	-0.0626	1.0000
7	6	4	6	-1.0000	0.0466	-1.0000	0.1422	-1.0000	0.3334
7	6	4	5	-0.7633	0.2333	-0.8297	0.3253	-0.9213	0.4861
7	6	4	4	-0.5417	0.3739	-0.6171	0.4554	-0.7416	0.5939

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N1	N2	X1	X2	90%		95%		99%	
7	6	4	3	-0.3860	0.4981	-0.4660	0.5702	-0.6019	0.6887
7	6	4	2	-0.2617	0.6295	-0.3726	0.6938	-0.4808	0.7970
7	6	4	1	-0.1183	0.8116	-0.2186	0.8655	-0.3528	0.9385
7	6	4	0	0.0	1.0000	-0.0529	1.0000	-0.1847	1.0000
7	7	7	6	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
7	7	7	5	-0.1778	1.0000	-0.2931	1.0000	-0.4969	1.0000
7	7	7	4	0.0	1.0000	-0.1001	1.0000	-0.2936	1.0000
7	7	7	3	0.0	1.0000	0.0	1.0000	-0.1403	1.0000
7	7	7	2	0.0	1.0000	0.0	1.0000	-0.0048	1.0000
7	7	7	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	7	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	7	6	7	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
7	7	6	6	-0.6446	0.6446	-0.7399	0.7399	-0.8766	0.8766
7	7	6	5	-0.3615	0.7233	-0.4575	0.7998	-0.6230	0.9066
7	7	6	4	-0.1848	0.7790	-0.2781	0.8414	-0.4437	0.9269
7	7	6	3	-0.0779	0.8244	-0.1524	0.8749	-0.3020	0.9429
7	7	6	2	0.0	0.8660	-0.0096	0.9052	-0.1720	0.9571
7	7	6	1	0.0	0.9100	0.0	0.9368	-0.0413	0.9717
7	7	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
7	7	5	7	-1.0000	0.1778	-1.0000	0.2931	-1.0000	0.4969
7	7	5	6	-0.7233	0.3615	-0.7998	0.4575	-0.9066	0.6230
7	7	5	5	-0.4790	0.4790	-0.5617	0.5617	-0.7007	0.7007
7	7	5	4	-0.3174	0.5719	-0.4013	0.6431	-0.5467	0.7602
7	7	5	3	-0.2169	0.6552	-0.2829	0.7156	-0.4208	0.8120
7	7	5	2	-0.1365	0.7404	-0.1459	0.7891	-0.3014	0.8639
7	7	5	1	0.0	0.8660	-0.0096	0.9052	-0.1720	0.9571
7	7	5	0	0.0	1.0000	0.0	1.0000	-0.0048	1.0000

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N1	N2	X1	X2	90%		95%		99%	
7	7	4	7	-1.0000	0.0	-1.0000	0.1001	-1.0000	0.2936
7	7	4	6	-0.7790	0.1848	-0.8414	0.2781	-0.9269	0.4437
7	7	4	5	-0.5719	0.3174	-0.6431	0.4013	-0.7602	0.5467
7	7	4	4	-0.4292	0.4292	-0.5048	0.5048	-0.6322	0.6322
7	7	4	3	-0.3250	0.5363	-0.4173	0.6036	-0.5256	0.7137
7	7	4	2	-0.2169	0.6552	-0.2829	0.7156	-0.4208	0.8120
7	7	4	1	-0.0779	0.8244	-0.1524	0.8749	-0.3020	0.9429
7	7	4	0	0.0	1.0000	0.0	1.0000	-0.1403	1.0000
8	1	8	0	-0.0294	1.0000	-0.2129	1.0000	-0.5573	1.0000
8	1	7	1	-1.0000	0.7891	-1.0000	0.8499	-1.0000	0.9314
8	1	7	0	-0.2318	1.0000	-0.3994	1.0000	-0.6835	1.0000
8	1	6	1	-1.0000	0.6975	-1.0000	0.7815	-1.0000	0.8980
8	1	6	0	-0.3592	1.0000	-0.5101	1.0000	-0.7511	1.0000
8	1	5	1	-1.0000	0.6185	-1.0000	0.7208	-1.0000	0.8673
8	1	5	0	-0.4570	1.0000	-0.5917	1.0000	-0.7980	1.0000
8	1	4	1	-1.0000	0.5407	-1.0000	0.6595	-1.0000	0.8350
8	1	4	0	-0.5407	1.0000	-0.6595	1.0000	-0.8350	1.0000
8	2	8	1	-0.2000	1.0000	-0.3710	1.0000	-0.6653	1.0000
8	2	8	0	0.0	1.0000	0.0	1.0000	-0.1194	1.0000
8	2	7	2	-1.0000	0.7143	-1.0000	0.7942	-1.0000	0.9043
8	2	7	1	-0.3905	0.8470	-0.5354	0.8918	-0.7654	0.9510
8	2	7	0	0.0	1.0000	-0.0500	1.0000	-0.2895	1.0000
8	2	6	2	-1.0000	0.5154	-1.0000	0.6041	-1.0000	0.7412
8	2	6	1	-0.5029	0.7778	-0.6279	0.8410	-0.8175	0.9269
8	2	6	0	-0.0575	1.0000	-0.1800	1.0000	-0.4038	1.0000
8	2	5	2	-1.0000	0.3893	-1.0000	0.4904	-1.0000	0.6553
8	2	5	1	-0.5857	0.7164	-0.6940	0.7949	-0.8529	0.9044
8	2	5	0	-0.1716	1.0000	-0.2888	1.0000	-0.4953	1.0000
8	2	4	2	-1.0000	0.2791	-1.0000	0.3893	-1.0000	0.5765
8	2	4	1	-0.6543	0.6543	-0.7475	0.7475	-0.8806	0.8806
8	2	4	0	-0.2791	1.0000	-0.3893	1.0000	-0.5765	1.0000
8	3	8	2	-0.2951	1.0000	-0.4549	1.0000	-0.7180	1.0000
8	3	8	1	0.0	1.0000	-0.0148	1.0000	-0.2565	1.0000
8	3	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	3	7	3	-1.0000	0.6610	-1.0000	0.7536	-1.0000	0.8840
8	3	7	2	-0.4736	0.7906	-0.6040	0.8505	-0.8043	0.9314
8	3	7	1	-0.0003	0.8735	-0.1974	0.9109	-0.4143	0.9698
8	3	7	0	0.0	1.0000	0.0	1.0000	-0.1088	1.0000

N1	N2	X1	X2	90%		95%		99%	
8	3	6	3	-1.0000	0.4202	-1.0000	0.5177	-1.0000	0.6755
8	3	6	2	-0.5756	0.6923	-0.6858	0.6918	-0.8485	0.7993
8	3	6	1	-0.2115	0.8152	-0.3208	0.8684	-0.5161	0.9399
8	3	6	0	-0.0	1.0000	-0.0288	1.0000	-0.2318	1.0000
8	3	5	3	-1.0000	0.3234	-1.0000	0.3729	-1.0000	0.5428
8	3	5	2	-0.6491	0.5146	-0.7432	0.5965	-0.8783	0.7295
8	3	5	1	-0.3193	0.7628	-0.4204	0.8296	-0.5954	0.9213
8	3	5	0	-0.0577	1.0000	-0.1432	1.0000	-0.3375	1.0000
8	3	4	3	-1.0000	0.1788	-1.0000	0.2544	-1.0000	0.4378
8	3	4	2	-0.7092	0.4174	-0.7892	0.5095	-0.9014	0.6642
8	3	4	1	-0.4174	0.7092	-0.5095	0.7892	-0.6642	0.9014
8	3	4	0	-0.1788	1.0000	-0.2544	1.0000	-0.4378	1.0000
8	4	8	3	-0.3592	1.0000	-0.5101	1.0000	-0.7511	1.0000
8	4	8	2	0.0	1.0000	-0.1052	1.0000	-0.3372	1.0000
8	4	8	1	0.0	1.0000	0.0	1.0000	-0.0718	1.0000
8	4	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	4	7	4	-1.0000	0.6185	-1.0000	0.7208	-1.0000	0.8673
8	4	7	3	-0.5278	0.7497	-0.6479	0.8200	-0.8283	0.9167
8	4	7	2	-0.1723	0.8261	-0.2834	0.8764	-0.4849	0.9437
8	4	7	1	0.0	0.8896	-0.0411	0.9224	-0.2352	0.9651
8	4	7	0	0.0	1.0000	0.0	1.0000	-0.0011	1.0000
8	4	6	4	-1.0000	0.3514	-1.0000	0.4548	-1.0000	0.6267
8	4	6	3	-0.6221	0.5414	-0.7222	0.6197	-0.8674	0.7460
8	4	6	2	-0.2992	0.6769	-0.4004	0.7368	-0.5782	0.8292
8	4	6	1	-0.0696	0.8381	-0.1695	0.8851	-0.3510	0.9477
8	4	6	0	0.0	1.0000	0.0	1.0000	-0.1262	1.0000
8	4	5	4	-1.0000	0.2164	-1.0000	0.2933	-1.0000	0.4703
8	4	5	3	-0.6892	0.4227	-0.7739	0.4906	-0.8937	0.6297
8	4	5	2	-0.4011	0.5806	-0.4929	0.6525	-0.6497	0.7684
8	4	5	1	-0.1838	0.7915	-0.2793	0.8508	-0.4478	0.9314
8	4	5	0	0.0	1.0000	-0.1210	1.0000	-0.2383	1.0000
8	4	4	4	-1.0000	0.0681	-1.0000	0.2178	-1.0000	0.3490
8	4	4	3	-0.7434	0.2927	-0.8149	0.3830	-0.9141	0.5376
8	4	4	2	-0.4920	0.4920	-0.5742	0.5742	-0.7109	0.7109
8	4	4	1	-0.2927	0.7434	-0.3830	0.8149	-0.5376	0.9141
8	4	4	0	-0.0681	1.0000	-0.2178	1.0000	-0.3490	1.0000
8	5	8	4	-0.4068	1.0000	-0.5502	1.0000	-0.7744	1.0000
8	5	8	3	-0.0488	1.0000	-0.1700	1.0000	-0.3933	1.0000
8	5	8	2	0.0	1.0000	0.0	1.0000	-0.1535	1.0000
8	5	8	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	5	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000

N1	N2	X1	X2	90%		95%		99%	
8	5	7	5	-1.0000	0.5829	-1.0000	0.6930	-1.0000	0.8528
8	5	7	4	-0.5671	0.7166	-0.6792	0.7950	-0.8450	0.9044
8	5	7	3	-0.2369	0.7914	-0.3432	0.8507	-0.5330	0.9314
8	5	7	2	-0.0278	0.8478	-0.1277	0.8921	-0.3116	0.9510
8	5	7	1	0.0	0.9008	0.0	0.9303	-0.1257	0.9687
8	5	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	5	6	5	-1.0000	0.2970	-1.0000	0.4045	-1.0000	0.5870
8	5	6	4	-0.6553	0.4817	-0.7479	0.5662	-0.8806	0.7060
8	5	6	3	-0.3595	0.6047	-0.4546	0.6731	-0.6198	0.7829
8	5	6	2	-0.1591	0.7121	-0.2530	0.7658	-0.4212	0.8485
8	5	6	1	-0.0058	0.8541	-0.1003	0.8966	-0.2469	0.9531
8	5	6	0	0.0	1.0000	0.0	1.0000	-0.0626	1.0000
8	5	5	5	-1.0000	0.1426	-1.0000	0.2324	-1.0000	0.4147
8	5	5	4	-0.7176	0.3318	-0.7954	0.4183	-0.9044	0.5660
8	5	5	3	-0.4564	0.4833	-0.5416	0.5549	-0.6857	0.6776
8	5	5	2	-0.2701	0.6238	-0.3580	0.6891	-0.5114	0.7938
8	5	5	1	-0.1183	0.8116	-0.2186	0.8655	-0.3528	0.9385
8	5	5	0	0.0	1.0000	-0.0210	1.0000	-0.1986	1.0000
8	5	4	5	-1.0000	0.0094	-1.0000	0.1635	-1.0000	0.2832
8	5	4	4	-0.7675	0.2221	-0.8329	0.3506	-0.9228	0.4553
8	5	4	3	-0.5417	0.3739	-0.6171	0.4554	-0.7416	0.5939
8	5	4	2	-0.3739	0.5417	-0.4554	0.6171	-0.5939	0.7416
8	5	4	1	-0.2221	0.7675	-0.3506	0.8329	-0.4553	0.9228
8	5	4	0	-0.0094	1.0000	-0.1635	1.0000	-0.2832	1.0000
8	6	8	5	-0.4441	1.0000	-0.5811	1.0000	-0.7920	1.0000
8	6	8	4	-0.1010	1.0000	-0.2200	1.0000	-0.4358	1.0000
8	6	8	3	0.0	1.0000	-0.0136	1.0000	-0.2128	1.0000
8	6	8	2	0.0	1.0000	0.0	1.0000	-0.0430	1.0000
8	6	8	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	6	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	6	7	6	-1.0000	0.5520	-1.0000	0.6685	-1.0000	0.8399
8	6	7	5	-0.5973	0.6885	-0.7030	0.7736	-0.8575	0.8937
8	6	7	4	-0.2861	0.7631	-0.3884	0.8297	-0.5688	0.9213
8	6	7	3	-0.0930	0.8170	-0.1899	0.8695	-0.3658	0.9404
8	6	7	2	0.0	0.8629	-0.0476	0.9030	-0.2032	0.9561
8	6	7	1	0.0	0.9091	0.0	0.9362	-0.0483	0.9714
8	6	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	6	6	6	-1.0000	0.2519	-1.0000	0.3624	-1.0000	0.5532
8	6	6	5	-0.6807	0.4337	-0.7673	0.5228	-0.8904	0.6730
8	6	6	4	-0.4048	0.5508	-0.4951	0.6253	-0.6505	0.7478
8	6	6	3	-0.2224	0.6460	-0.3117	0.7079	-0.4703	0.8068
8	6	6	2	-0.1095	0.7374	-0.1677	0.7867	-0.3202	0.8623
8	6	6	1	0.0	0.8660	-0.0096	0.9052	-0.1720	0.9571
8	6	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000

N1	N2	X1	X2	90%		95%		99%	
8	6	5	6	-1.0000	0.0856	-1.0000	0.1830	-1.0000	0.3692
8	6	5	5	-0.7391	0.2732	-0.8116	0.3623	-0.9124	0.5167
8	6	5	4	-0.4974	0.4101	-0.5775	0.4881	-0.7120	0.6197
8	6	5	3	-0.3300	0.5297	-0.4124	0.5980	-0.5553	0.7096
8	6	5	2	-0.2169	0.6552	-0.2829	0.7156	-0.4208	0.8120
8	6	5	1	-0.0934	0.8267	-0.1230	0.8766	-0.2833	0.9437
8	6	5	0	0.0	1.0000	0.0	1.0000	-0.1112	1.0000
8	6	4	6	-1.0000	0.0	-1.0000	0.0960	-1.0000	0.2306
8	6	4	5	-0.7857	0.1722	-0.8464	0.2595	-0.9293	0.3940
8	6	4	4	-0.5702	0.3112	-0.6485	0.4093	-0.7639	0.5169
8	6	4	3	-0.4292	0.4292	-0.5048	0.5048	-0.6322	0.6322
8	6	4	2	-0.3112	0.5782	-0.4093	0.6485	-0.5169	0.7639
8	6	4	1	-0.1722	0.7857	-0.2595	0.8464	-0.3940	0.9293
8	6	4	0	0.0	1.0000	-0.0960	1.0000	-0.2306	1.0000
8	7	8	6	-0.4745	1.0000	-0.6061	1.0000	-0.8060	1.0000
8	7	8	5	-0.1437	1.0000	-0.2605	1.0000	-0.4695	1.0000
8	7	8	4	0.0	1.0000	-0.0640	1.0000	-0.2590	1.0000
8	7	8	3	0.0	1.0000	0.0	1.0000	-0.1023	1.0000
8	7	8	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	7	8	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	7	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	7	7	7	-1.0000	0.5246	-1.0000	0.6466	-1.0000	0.8281
8	7	7	6	-0.6216	0.6638	-0.7220	0.7547	-0.8674	0.8841
8	7	7	5	-0.3256	0.7389	-0.4244	0.8115	-0.5968	0.9124
8	7	7	4	-0.1438	0.7918	-0.2380	0.8509	-0.4074	0.9315
8	7	7	3	-0.0461	0.8340	-0.1003	0.8825	-0.2594	0.9465
8	7	7	2	0.0	0.8741	0.0	0.9111	-0.1259	0.9598
8	7	7	1	0.0	0.9155	0.0	0.9408	-0.0013	0.9735
8	7	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	7	6	7	-1.0000	0.2133	-1.0000	0.3262	-1.0000	0.5237
8	7	6	6	-0.7009	0.3933	-0.7828	0.4861	-0.8982	0.6447
8	7	6	5	-0.4406	0.5071	-0.5270	0.5863	-0.6743	0.7187
8	7	6	4	-0.2709	0.5963	-0.3566	0.6642	-0.5074	0.7751
8	7	6	3	-0.1895	0.6759	-0.2186	0.7331	-0.3724	0.8240
8	7	6	2	-0.0686	0.7567	-0.0973	0.8026	-0.2468	0.8728
8	7	6	1	0.0	0.8754	0.0	0.9119	-0.1217	0.9602
8	7	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	7	5	7	-1.0000	0.0391	-1.0000	0.1414	-1.0000	0.3304
8	7	5	6	-0.7562	0.2256	-0.8244	0.3163	-0.9186	0.4758
8	7	5	5	-0.5296	0.3553	-0.6055	0.4359	-0.7323	0.5746
8	7	5	4	-0.3753	0.4635	-0.4535	0.5354	-0.5882	0.6561
8	7	5	3	-0.2888	0.5662	-0.3309	0.6298	-0.4686	0.7332
8	7	5	2	-0.1880	0.6795	-0.2076	0.7360	-0.3540	0.8260
8	7	5	1	-0.0529	0.8386	-0.0735	0.8852	-0.2283	0.9478
8	7	5	0	0.0	1.0000	0.0	1.0000	-0.0636	1.0000

N1	N2	X1	X2	90%		95%		99%	
8	7	4	7	-1.0000	0.0	-1.0000	0.0433	-1.0000	0.1868
8	7	4	6	-0.8001	0.1352	-0.8570	0.1944	-0.9344	0.3447
8	7	4	5	-0.6066	0.2718	-0.6728	0.3223	-0.7810	0.4593
8	7	4	4	-0.4706	0.3730	-0.5416	0.4524	-0.6607	0.5595
8	7	4	3	-0.3730	0.4706	-0.4524	0.5416	-0.5595	0.6607
8	7	4	2	-0.2718	0.6066	-0.3223	0.6728	-0.4593	0.7810
8	7	4	1	-0.1352	0.8001	-0.1944	0.8570	-0.3447	0.9344
8	7	4	0	0.0	1.0000	-0.0433	1.0000	-0.1868	1.0000
8	8	8	7	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
8	8	8	6	-0.1796	1.0000	-0.2943	1.0000	-0.4973	1.0000
8	8	8	5	-0.0000	1.0000	-0.1055	1.0000	-0.2966	1.0000
8	8	8	4	0.0	1.0000	0.0	1.0000	-0.1493	1.0000
8	8	8	3	0.0	1.0000	0.0	1.0000	-0.0244	1.0000
8	8	8	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	8	8	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	8	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	8	7	8	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
8	8	7	7	-0.6417	0.6417	-0.7376	0.7376	-0.8754	0.8754
8	8	7	6	-0.3583	0.7176	-0.4540	0.7954	-0.6197	0.9044
8	8	7	5	-0.1853	0.7702	-0.2771	0.8348	-0.4407	0.9237
8	8	7	4	-0.1087	0.8119	-0.1429	0.8656	-0.3032	0.9385
8	8	7	3	0.0	0.8482	-0.0332	0.8922	-0.1829	0.9510
8	8	7	2	0.0	0.8830	0.0	0.9174	-0.0886	0.9628
8	8	7	1	0.0	0.9208	0.0	0.9445	0.0	0.9752
8	8	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	8	6	8	-1.0000	0.1796	-1.0000	0.2943	-1.0000	0.4973
8	8	6	7	-0.7176	0.3583	-0.7954	0.4540	-0.9044	0.6197
8	8	6	6	-0.4700	0.4700	-0.5530	0.5530	-0.6936	0.6936
8	8	6	5	-0.3100	0.5558	-0.3926	0.6283	-0.5370	0.7487
8	8	6	4	-0.2595	0.6295	-0.2647	0.6925	-0.4126	0.7948
8	8	6	3	-0.1210	0.6990	-0.1613	0.7524	-0.3008	0.8372
8	8	6	2	0.0	0.7722	-0.0562	0.8153	-0.2024	0.8812
8	8	6	1	0.0	0.8830	0.0	0.9174	-0.0886	0.9628
8	8	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
8	8	5	8	-1.0000	0.0000	-1.0000	0.1055	-1.0000	0.2966
8	8	5	7	-0.7702	0.1853	-0.8348	0.2771	-0.9237	0.4407
8	8	5	6	-0.5558	0.3100	-0.6283	0.3926	-0.7487	0.5370
8	8	5	5	-0.4114	0.4114	-0.4862	0.4862	-0.6142	0.6142
8	8	5	4	-0.3469	0.5033	-0.3696	0.5706	-0.5049	0.6831
8	8	5	3	-0.2386	0.5947	-0.2680	0.6546	-0.4042	0.7516
8	8	5	2	-0.1210	0.6990	-0.1613	0.7524	-0.3008	0.8372
8	8	5	1	0.0	0.8482	-0.0332	0.8922	-0.1829	0.9510
8	8	5	0	0.0	1.0000	0.0	1.0000	-0.0244	1.0000
8	8	4	8	-1.0000	0.0	-1.0000	0.0	-1.0000	0.1493
8	8	4	7	-0.8119	0.1087	-0.8656	0.1429	-0.9385	0.3032
8	8	4	6	-0.6295	0.2595	-0.6925	0.2647	-0.7948	0.4126

N1	N2	X1	X2	90%		95%		99%	
~~8~~	~~8~~	~~4~~	~~5~~	~~-0.5033~~	~~0.3469~~	~~-0.5706~~	~~0.3696~~	~~-0.6831~~	~~0.5049~~
8	8	4	4	-0.4197	0.4197	-0.4862	0.4862	-0.5916	0.5916
~~8~~	~~8~~	~~4~~	~~3~~	~~-0.3469~~	~~0.5033~~	~~-0.3696~~	~~0.5706~~	~~-0.5049~~	~~0.6831~~
8	8	4	2	-0.2595	0.6295	-0.2647	0.6925	-0.4126	0.7948
~~8~~	~~8~~	~~4~~	~~1~~	~~-0.1047~~	~~0.8119~~	~~-0.1429~~	~~0.8656~~	~~-0.3032~~	~~0.9385~~
8	8	4	0	0.0	1.0000	0.0	1.0000	-0.1493	1.0000
9	1	9	0	-0.0000	1.0000	-0.1847	1.0000	-0.5367	1.0000
9	1	8	1	-1.0000	0.8000	-1.0000	0.8579	-1.0000	0.9352
9	1	8	0	-0.2000	1.0000	-0.3710	1.0000	-0.6655	1.0000
9	1	7	1	-1.0000	0.7143	-1.0000	0.7942	-1.0000	0.9043
9	1	7	0	-0.3252	1.0000	-0.4810	1.0000	-0.7338	1.0000
9	1	6	1	-1.0000	0.6417	-1.0000	0.7388	-1.0000	0.8765
9	1	6	0	-0.4202	1.0000	-0.5615	1.00[illegible]	-0.7808	1.0000
9	1	5	1	-1.0000	0.5721	-1.0000	0.6845	-1.0000	0.8483
9	1	5	0	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
9	2	9	1	-0.1716	1.0000	-0.3453	1.0000	-0.6485	1.0000
9	2	9	0	0.0	1.0000	0.0	1.0000	-0.0887	1.0000
9	2	8	2	-1.0000	0.7284	-1.0000	0.8048	-1.0000	0.9095
9	2	8	1	-0.3619	0.8551	-0.5111	0.8977	-0.7512	0.9537
9	2	8	0	0.0	1.0000	-0.0148	1.0000	-0.2565	1.0000
9	2	7	2	-1.0000	0.5387	-1.0000	0.6240	-1.0000	0.7551
9	2	7	1	-0.4736	0.7906	-0.6040	0.8505	-0.8043	0.9314
9	2	7	0	-0.0169	1.0000	-0.1399	1.0000	-0.3684	1.0000
9	2	6	2	-1.0000	0.4202	-1.0000	0.5177	-1.0000	0.6755
9	2	6	1	-0.5550	0.7347	-0.6697	0.8087	-0.8400	0.9112
9	2	6	0	-0.1239	1.0000	-0.2431	1.0000	-0.4566	1.0000
9	2	5	2	-1.0000	0.3186	-1.0000	0.4252	-1.0000	0.6042
9	2	5	1	-0.6213	0.6797	-0.7219	0.7670	-0.8674	0.8904
9	2	5	0	-0.2221	1.0000	-0.3356	1.0000	-0.5330	1.0000
9	3	9	2	-0.2679	1.0000	-0.4313	1.0000	-0.7035	1.0000
9	3	9	1	0.0	1.0000	0.0	1.0000	-0.2272	1.0000
9	3	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	3	8	3	-1.0000	0.6772	-1.0000	0.7661	-1.0000	0.8903
9	3	8	2	-0.4474	0.8014	-0.5824	0.8584	-0.7922	0.9352
9	3	8	1	-0.0455	0.8803	-0.1634	0.9157	-0.3843	0.9621
9	3	8	0	0.0	1.0000	0.0	1.0000	-0.0718	1.0000
9	3	7	3	-1.0000	0.4464	-1.0000	0.5408	-1.0000	0.6924
9	3	7	2	-0.5493	0.6417	-0.6649	0.7082	-0.8373	0.8104
9	3	7	1	-0.1723	0.8261	-0.2834	0.8764	-0.4849	0.9437
9	3	7	0	0.0	1.0000	0.0	1.0000	-0.1898	1.0000

N1	N2	X1	X2	90%		95%		99%	
9	3	6	3	-1.0000	0.3533	-1.0000	0.4036	-1.0000	0.5672
9	3	6	2	-0.6221	0.5414	-0.7222	0.6197	-0.8674	0.7460
9	3	6	1	-0.2749	0.7785	-0.3790	0.8412	-0.5622	0.9269
9	3	6	0	0.0	1.0000	-0.0924	1.0000	-0.2891	1.0000
9	3	5	3	-1.0000	0.2164	-1.0000	0.2933	-1.0000	0.4703
9	3	5	2	-0.6804	0.4527	-0.7673	0.5407	-0.8904	0.6873
9	3	5	1	-0.3660	0.7312	-0.4624	0.8058	-0.6276	0.9096
9	3	5	0	-0.1040	1.0000	-0.1924	1.0000	-0.3803	1.0000
9	4	9	3	-0.3333	1.0000	-0.4880	1.0000	-0.7380	1.0000
9	4	9	2	0.0	1.0000	-0.0746	1.0000	-0.3093	1.0000
9	4	9	1	0.0	1.0000	0.0	1.0000	-0.0396	1.0000
9	4	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	4	8	4	-1.0000	0.6364	-1.0000	0.7347	-1.0000	0.8744
9	4	8	3	-0.5034	0.7623	-0.6281	0.8294	-0.8175	0.9212
9	4	8	2	-0.1382	0.8353	-0.2507	0.8830	-0.4572	0.9468
9	4	8	1	0.0	0.8956	-0.0048	0.9267	-0.1996	0.9671
9	4	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	4	7	4	-1.0000	0.3793	-1.0000	0.4798	-1.0000	0.6456
9	4	7	3	-0.5979	0.5638	-0.7032	0.6390	-0.8576	0.7598
9	4	7	2	-0.2617	0.6941	-0.3653	0.7511	-0.5498	0.8388
9	4	7	1	-0.0278	0.8478	-0.1277	0.8921	-0.3116	0.9510
9	4	7	0	0.0	1.0000	0.0	1.0000	-0.0813	1.0000
9	4	6	4	-1.0000	0.2484	-1.0000	0.3258	-1.0000	0.4973
9	4	6	3	-0.6646	0.4500	-0.7550	0.5175	-0.8842	0.6505
9	4	6	2	-0.3595	0.6047	-0.4546	0.6731	-0.6198	0.7829
9	4	6	1	-0.1345	0.8055	-0.2308	0.8611	-0.4036	0.9364
9	4	6	0	0.0	1.0000	-0.0529	1.0000	-0.1847	1.0000
9	4	5	4	-1.0000	0.1108	-1.0000	0.2692	-1.0000	0.3840
9	4	5	3	-0.7176	0.3313	-0.7954	0.4183	-0.9044	0.5660
9	4	5	2	-0.4447	0.5244	-0.5314	0.6024	-0.6784	0.7313
9	4	5	1	-0.2333	0.7633	-0.3253	0.8297	-0.4861	0.9213
9	4	5	0	-0.0094	1.0000	-0.1633	1.0000	-0.2832	1.0000
9	5	9	4	-0.3820	1.0000	-0.5293	1.0000	-0.7624	1.0000
9	5	9	3	-0.0181	1.0000	-0.1398	1.0000	-0.3667	1.0000
9	5	9	2	0.0	1.0000	0.0	1.0000	-0.1217	1.0000
9	5	9	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	5	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	5	8	5	-1.0000	0.6020	-1.0000	0.7079	-1.0000	0.8606
9	5	8	4	-0.5441	0.7307	-0.6608	0.8056	-0.8352	0.9096
9	5	8	3	-0.2036	0.8021	-0.3118	0.8587	-0.5070	0.9352
9	5	8	2	0.0	0.8559	-0.0917	0.8980	-0.2774	0.9538
9	5	8	1	0.0	0.9062	0.0	0.9342	-0.0875	0.9705
9	5	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000

N1	N2	X1	X2	90%		95%		99%	
9	5	7	5	-1.0000	0.3260	-1.0000	0.4309	-1.0000	0.6074
9	5	7	4	-0.6328	0.5061	-0.7304	0.5875	-0.8716	0.7215
9	5	7	3	-0.3236	0.6248	-0.4214	0.6902	-0.5935	0.7949
9	5	7	2	-0.1179	0.7277	-0.2124	0.7788	-0.3841	0.8571
9	5	7	1	0.0	0.8629	-0.0476	0.9030	-0.2032	0.9561
9	5	7	0	0.0	1.0000	0.0	1.0000	-0.0048	1.0000
9	5	6	5	-1.0000	0.1755	-1.0000	0.2661	-1.0000	0.4433
9	5	6	4	-0.6948	0.3634	-0.7781	0.4475	-0.8958	0.5894
9	5	6	3	-0.4170	0.5088	-0.5058	0.5794	-0.6583	0.6962
9	5	6	2	-0.2224	0.6460	-0.3117	0.7079	-0.4703	0.8068
9	5	6	1	-0.0779	0.8244	-0.1524	0.8749	-0.3020	0.9429
9	5	6	0	0.0	1.0000	0.0	1.0000	-0.1112	1.0000
9	5	5	5	-1.0000	0.0529	-1.0000	0.1990	-1.0000	0.3197
9	5	5	4	-0.7438	0.2633	-0.8151	0.3624	-0.9141	0.4866
9	5	5	3	-0.4974	0.4101	-0.5775	0.4881	-0.7120	0.6197
9	5	5	2	-0.3174	0.5719	-0.4013	0.6431	-0.5467	0.7602
9	5	5	1	-0.1722	0.7857	-0.2595	0.8464	-0.3940	0.9293
9	5	5	0	0.0	1.0000	-0.0643	1.0000	-0.2372	1.0000
9	6	9	5	-0.4202	1.0000	-0.5613	1.0000	-0.7808	1.0000
9	6	9	4	-0.0705	1.0000	-0.1904	1.0000	-0.4102	1.0000
9	6	9	3	0.0	1.0000	0.0	1.0000	-0.1816	1.0000
9	6	9	2	0.0	1.0000	0.0	1.0000	-0.0098	1.0000
9	6	9	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	6	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	6	8	6	-1.0000	0.5721	-1.0000	0.6845	-1.0000	0.8483
9	6	8	5	-0.5755	0.7037	-0.6857	0.7852	-0.8485	0.8995
9	6	8	4	-0.2537	0.7751	-0.3581	0.8386	-0.5441	0.9256
9	6	8	3	-0.0573	0.8266	-0.1544	0.8765	-0.3328	0.9437
9	6	8	2	0.0	0.8702	-0.0043	0.9083	-0.1657	0.9586
9	6	8	1	0.0	0.9140	0.0	0.9397	-0.0089	0.9730
9	6	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	6	7	6	-1.0000	0.2817	-1.0000	0.3698	-1.0000	0.5748
9	6	7	5	-0.6595	0.4595	-0.7511	0.5457	-0.8822	0.6900
9	6	7	4	-0.3702	0.5729	-0.4635	0.6444	-0.6258	0.7614
9	6	7	3	-0.1820	0.6644	-0.2724	0.7235	-0.4350	0.8176
9	6	7	2	-0.0831	0.7518	-0.1158	0.7986	-0.2780	0.8702
9	6	7	1	0.0	0.8741	0.0	0.9111	-0.1259	0.9598
9	6	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	6	6	6	-1.0000	0.1190	-1.0000	0.2174	-1.0000	0.3990
9	6	6	5	-0.7178	0.3065	-0.7955	0.3930	-0.9044	0.5419
9	6	6	4	-0.4599	0.4400	-0.5437	0.5149	-0.6864	0.6408
9	6	6	3	-0.2839	0.5555	-0.3682	0.6207	-0.5165	0.7267
9	6	6	2	-0.1895	0.6759	-0.2186	0.7331	-0.3724	0.8240
9	6	6	1	-0.0529	0.8386	-0.0735	0.8852	-0.2283	0.9478
9	6	6	0	0.0	1.0000	0.0	1.0000	-0.0561	1.0000

N1	N2	X1	X2	90%		95%		99%	
9	6	5	6	-1.0000	0.0070	-1.0000	0.1322	-1.0000	0.2682
9	6	5	5	-0.7636	0.2162	-0.8298	0.2934	-0.9213	0.4273
9	6	5	4	-0.5364	0.3509	-0.6115	0.4394	-0.7364	0.5457
9	6	5	3	-0.3753	0.4635	-0.4535	0.5354	-0.5882	0.6561
9	6	5	2	-0.2718	0.6066	-0.3223	0.6728	-0.4593	0.7810
9	6	5	1	-0.1635	0.8027	-0.1722	0.8589	-0.3262	0.9353
9	6	5	0	0.0	1.0000	0.0	1.0000	-0.1564	1.0000
9	7	9	6	-0.4514	1.0000	-0.5871	1.0000	-0.7954	1.0000
9	7	9	5	-0.1135	1.0000	-0.2314	1.0000	-0.4449	1.0000
9	7	9	4	0.0	1.0000	-0.0323	1.0000	-0.2284	1.0000
9	7	9	3	0.0	1.0000	0.0	1.0000	-0.0692	1.0000
9	7	9	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	7	9	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	7	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	7	8	7	-1.0000	0.5456	-1.0000	0.6634	-1.0000	0.8372
9	7	8	6	-0.6008	0.6800	-0.7057	0.7671	-0.8589	0.8904
9	7	8	5	-0.2940	0.7520	-0.3951	0.8213	-0.5734	0.9172
9	7	8	4	-0.1084	0.8026	-0.2032	0.8588	-0.3755	0.9353
9	7	8	3	-0.0213	0.8436	-0.0572	0.8889	-0.2228	0.9495
9	7	8	2	0.0	0.8809	0.0	0.9159	-0.0867	0.9621
9	7	8	1	0.0	0.9202	0.0	0.9441	0.0	0.9750
9	7	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	7	7	7	-1.0000	0.2437	-1.0000	0.3544	-1.0000	0.5463
9	7	7	6	-0.6808	0.4202	-0.7674	0.5101	-0.8904	0.6628
9	7	7	5	-0.4072	0.5307	-0.4967	0.6069	-0.6510	0.7337
9	7	7	4	-0.2314	0.6168	-0.3185	0.6817	-0.4737	0.7874
9	7	7	3	-0.1677	0.6931	-0.1755	0.7475	-0.3317	0.8339
9	7	7	2	-0.0157	0.7702	-0.0579	0.8137	-0.2016	0.8801
9	7	7	1	0.0	0.8830	0.0	0.9174	-0.0886	0.9628
9	7	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	7	6	7	-1.0000	0.0728	-1.0000	0.1762	-1.0000	0.3612
9	7	6	6	-0.7361	0.2598	-0.8092	0.3481	-0.9112	0.5024
9	7	6	5	-0.4937	0.3866	-0.5734	0.4644	-0.7082	0.5975
9	7	6	4	-0.3307	0.4917	-0.4111	0.5605	-0.5514	0.6756
9	7	6	3	-0.2684	0.5906	-0.2767	0.6510	-0.4223	0.7491
9	7	6	2	-0.1210	0.6990	-0.1613	0.7524	-0.3008	0.8372
9	7	6	1	0.0	0.8497	-0.0297	0.8933	-0.1816	0.9515
9	7	6	0	0.0	1.0000	0.0	1.0000	-0.0152	1.0000
9	7	5	7	-1.0000	0.0	-1.0000	0.0799	-1.0000	0.2251
9	7	5	6	-0.7792	0.1828	-0.8415	0.2294	-0.9269	0.3793
9	7	5	5	-0.5668	0.3168	-0.6377	0.3548	-0.7553	0.4902
9	7	5	4	-0.4189	0.4113	-0.4928	0.4811	-0.6192	0.5864
9	7	5	3	-0.3469	0.5033	-0.3696	0.5706	-0.5049	0.6851
9	7	5	2	-0.2294	0.6335	-0.2555	0.6958	-0.3945	0.7971
9	7	5	1	-0.0960	0.8160	-0.1240	0.8687	-0.2724	0.9399
9	7	5	0	0.0	1.0000	0.0	1.0000	-0.1104	1.0000

N1	N2	X1	X2	90%		95%		99%	
9	8	9	7	-0.4776	1.0000	-0.6086	1.0000	-0.8075	1.0000
9	8	9	6	-0.1497	1.0000	-0.2658	1.0000	-0.4735	1.0000
9	8	9	5	0.0	1.0000	-0.0739	1.0000	-0.2666	1.0000
9	8	9	4	0.0	1.0000	0.0	1.0000	-0.1165	1.0000
9	8	9	3	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	8	9	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	8	9	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	8	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	8	8	8	-1.0000	0.5218	-1.0000	0.6443	-1.0000	0.8269
9	8	8	7	-0.6217	0.6588	-0.7221	0.7508	-0.8674	0.8822
9	8	8	6	-0.3275	0.7316	-0.4256	0.8060	-0.5972	0.9096
9	8	8	5	-0.1503	0.7819	-0.2430	0.8435	-0.4099	0.9279
9	8	8	4	-0.1003	0.8217	-0.1039	0.8728	-0.2674	0.9419
9	8	8	3	0.0	0.8563	0.0	0.8981	-0.1441	0.9538
9	8	8	2	0.0	0.8893	0.0	0.9220	-0.0809	0.9649
9	8	8	1	0.0	0.9252	0.0	0.9476	0.0	0.9766
9	8	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	8	7	8	-1.0000	0.2105	-1.0000	0.3231	-1.0000	0.5208
9	8	7	7	-0.6984	0.3861	-0.7808	0.4791	-0.8972	0.6388
9	8	7	6	-0.4377	0.4948	-0.5238	0.5749	-0.6714	0.7097
9	8	7	5	-0.2714	0.5777	-0.3556	0.6473	-0.5046	0.7623
9	8	7	4	-0.2094	0.6486	-0.2231	0.7088	-0.3732	0.8062
9	8	7	3	-0.0686	0.7152	-0.1242	0.7660	-0.2566	0.8464
9	8	7	2	0.0	0.7849	-0.0331	0.8258	-0.1941	0.8881
9	8	7	1	0.0	0.8901	0.0	0.9225	-0.0577	0.9651
9	8	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	8	6	8	-1.0000	0.0362	-1.0000	0.1407	-1.0000	0.3281
9	8	6	7	-0.7510	0.2201	-0.8205	0.3098	-0.9167	0.4684
9	8	6	6	-0.5212	0.3425	-0.5975	0.4225	-0.7259	0.5614
9	8	6	5	-0.3682	0.4412	-0.4452	0.5131	-0.5791	0.6355
9	8	6	4	-0.3081	0.5300	-0.3219	0.5943	-0.4605	0.7013
9	8	6	3	-0.1728	0.6179	-0.2249	0.6747	-0.3528	0.7665
9	8	6	2	-0.0529	0.7175	-0.1278	0.7680	-0.2628	0.8477
9	8	6	1	0.0	0.8587	0.0	0.8999	-0.1608	0.9546
9	8	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	8	5	8	-1.0000	0.0	-1.0000	0.0367	-1.0000	0.1880
9	8	5	7	-0.7921	0.1662	-0.8510	0.1809	-0.9315	0.3389
9	8	5	6	-0.5915	0.2934	-0.6590	0.3026	-0.7704	0.4451
9	8	5	5	-0.4535	0.4002	-0.5239	0.4048	-0.6437	0.5340
9	8	5	4	-0.4057	0.4567	-0.4118	0.5137	-0.5395	0.6170
9	8	5	3	-0.2790	0.5346	-0.3146	0.5982	-0.4428	0.7043
9	8	5	2	-0.1635	0.6552	-0.2114	0.7143	-0.3428	0.8099
9	8	5	1	-0.0433	0.8269	-0.0848	0.8767	-0.2277	0.9437
9	8	5	0	0.0	1.0000	0.0	1.0000	-0.0714	1.0000
9	9	9	8	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
9	9	9	7	-0.1810	1.0000	-0.2952	1.0000	-0.4977	1.0000

N1	N2	X1	X2	90%		95%		99%	
9	9	9	6	-0.0048	1.0000	-0.1093	1.0000	-0.2988	1.0000
~~9~~	~~9~~	~~9~~	~~5~~	0.0	1.0000	0.0	1.0000	-0.1556	1.0000
9	9	9	4	0.0	1.0000	0.0	1.0000	-0.0375	1.0000
9	9	9	3	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	9	9	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	9	9	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	9	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	9	8	9	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
9	9	8	8	-0.6395	0.6395	-0.7359	0.7359	-0.8745	0.8745
9	9	8	7	-0.3560	0.7132	-0.4514	0.7921	-0.6172	0.9028
9	9	8	6	-0.1856	0.7636	-0.2763	0.8299	-0.4386	0.9213
9	9	8	5	-0.1365	0.8028	-0.1445	0.8589	-0.3038	0.9353
9	9	8	4	0.0	0.8361	-0.0520	0.8833	-0.1895	0.9468
9	9	8	3	0.0	0.8663	0.0	0.9053	-0.1262	0.9571
9	9	8	2	0.0	0.8961	0.0	0.9268	0.0	0.9671
9	9	8	1	0.0	0.9293	0.0	0.9505	0.0	0.9779
9	9	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	9	7	9	-1.0000	0.1810	-1.0000	0.2952	-1.0000	0.4977
9	9	7	8	-0.7132	0.3560	-0.7921	0.4514	-0.9028	0.6172
9	9	7	7	-0.4635	0.4635	-0.5467	0.5467	-0.6884	0.6884
9	9	7	6	-0.3047	0.5444	-0.3864	0.6177	-0.5302	0.7405
9	9	7	5	-0.2443	0.6121	-0.2618	0.6767	-0.4069	0.7830
9	9	7	4	-0.1115	0.6733	-0.1768	0.7296	-0.2997	0.8206
9	9	7	3	0.0	0.7328	-0.1112	0.7808	-0.2377	0.8564
9	9	7	2	0.0	0.7970	0.0	0.8357	-0.1150	0.8946
9	9	7	1	0.0	0.8961	0.0	0.9268	0.0	0.9671
9	9	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	9	6	9	-1.0000	0.0048	-1.0000	0.1093	-1.0000	0.2988
9	9	6	8	-0.7636	0.1856	-0.8299	0.2763	-0.9213	0.4386
9	9	6	7	-0.5444	0.3047	-0.6177	0.3864	-0.7405	0.5302
9	9	6	6	-0.3992	0.3992	-0.4735	0.4735	-0.6018	0.6018
9	9	6	5	-0.3410	0.4820	-0.3583	0.5494	-0.4912	0.6637
9	9	6	4	-0.2145	0.5601	-0.2746	0.6208	-0.3931	0.7215
9	9	6	3	-0.1055	0.6400	-0.2185	0.6938	-0.3252	0.7806
9	9	6	2	0.0	0.7328	-0.1112	0.7808	-0.2377	0.8564
9	9	6	1	0.0	0.8663	0.0	0.9053	-0.1262	0.9571
9	9	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
9	9	5	9	-1.0000	0.0	-1.0000	0.0	-1.0000	0.1556
9	9	5	8	-0.8028	0.1365	-0.8589	0.1445	-0.9353	0.3038
9	9	5	7	-0.6121	0.2443	-0.6767	0.2618	-0.7830	0.4069
9	9	5	6	-0.4820	0.3410	-0.5494	0.3583	-0.6637	0.4912
9	9	5	5	-0.4359	0.4359	-0.4455	0.4455	-0.5671	0.5671
9	9	5	4	-0.3184	0.4927	-0.3599	0.5403	-0.4798	0.6412
9	9	5	3	-0.2145	0.5601	-0.2746	0.6208	-0.3931	0.7215
9	9	5	2	-0.1115	0.6733	-0.1768	0.7296	-0.2997	0.8206
9	9	5	1	0.0	0.8361	-0.0520	0.8833	-0.1895	0.9468
9	9	5	0	0.0	1.0000	0.0	1.0000	-0.0375	1.0000

N1	N2	X1	X2	90%		95%		99%	
10	1	10	0	0.0	1.0000	-0.1591	1.0000	-0.5177	1.0000
10	1	9	1	-1.0000	0.8093	-1.0000	0.8647	-1.0000	0.9384
10	1	9	0	-0.1716	1.0000	-0.3453	1.0000	-0.6485	1.0000
10	1	8	1	-1.0000	0.7284	-1.0000	0.8048	-1.0000	0.9095
10	1	8	0	-0.2951	1.0000	-0.4549	1.0000	-0.7180	1.0000
10	1	7	1	-1.0000	0.6610	-1.0000	0.7536	-1.0000	0.8840
10	1	7	0	-0.3880	1.0000	-0.5343	1.0000	-0.7653	1.0000
10	1	6	1	-1.0000	0.5975	-1.0000	0.7044	-1.0000	0.8588
10	1	6	0	-0.4650	1.0000	-0.5983	1.0000	-0.8016	1.0000
~~10~~	~~1~~	~~5~~	~~1~~	-1.0000	0.5334	-1.0000	0.6537	-1.0000	0.8319
10	1	5	0	-0.5334	1.0000	-0.6537	1.0000	-0.8319	1.0000
10	2	10	1	-0.1459	1.0000	-0.3219	1.0000	-0.6330	1.0000
10	2	10	0	0.0	1.0000	0.0	1.0000	-0.0612	1.0000
10	2	9	2	-1.0000	0.7405	-1.0000	0.8139	-1.0000	0.9140
10	2	9	1	-0.3361	0.8621	-0.4891	0.9027	-0.7381	0.9561
10	2	9	0	0.0	1.0000	0.0	1.0000	-0.2272	1.0000
10	2	8	2	-1.0000	0.5587	-1.0000	0.6411	-1.0000	0.7670
10	2	8	1	-0.4474	0.8014	-0.5824	0.8584	-0.7922	0.9352
10	2	8	0	0.0	1.0000	-0.1052	1.0000	-0.3372	1.0000
10	2	7	2	-1.0000	0.4464	-1.0000	0.5408	-1.0000	0.6924
10	2	7	1	-0.5278	0.7497	-0.6479	0.8200	-0.8283	0.9167
10	2	7	0	-0.0836	1.0000	-0.2040	1.0000	-0.4230	1.0000
10	2	6	2	-1.0000	0.3514	-1.0000	0.4548	-1.0000	0.6267
10	2	6	1	-0.5926	0.6999	-0.6994	0.7824	-0.8557	0.8981
10	2	6	0	-0.1753	1.0000	-0.2914	1.0000	-0.4962	1.0000
10	2	5	2	-1.0000	0.2630	-1.0000	0.3734	-1.0000	0.5628
10	2	5	1	-0.6486	0.6486	-0.7430	0.7430	-0.8783	0.8783
10	2	5	0	-0.2630	1.0000	-0.3734	1.0000	-0.5628	1.0000
10	3	10	2	-0.2433	1.0000	-0.4096	1.0000	-0.6899	1.0000
10	3	10	1	0.0	1.0000	0.0	1.0000	-0.2008	1.0000
10	3	10	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	3	9	3	-1.0000	0.6912	-1.0000	0.7767	-1.0000	0.8956
10	3	9	2	-0.4235	0.8107	-0.5626	0.8652	-0.7809	0.9384
10	3	9	1	-0.0149	0.8861	-0.1332	0.9199	-0.3574	0.9640
10	3	9	0	0.0	1.0000	0.0	1.0000	-0.0396	1.0000
10	3	8	3	-1.0000	0.4689	-1.0000	0.5605	-1.0000	0.7067
10	3	8	2	-0.5256	0.6583	-0.6458	0.7220	-0.8270	0.8199

N1	N2	X1	X2	90%		95%		99%	
10	3	9	1	-0.1382	0.8353	-0.2507	0.8830	-0.4572	0.9468
10	3	8	0	0.0	1.0000	0.0	1.0000	-0.1535	1.0000
10	3	7	3	-1.0000	0.3791	-1.0000	0.4296	-1.0000	0.5878
10	3	7	2	-0.5979	0.5638	-0.7032	0.6390	-0.8576	0.7598
10	3	7	1	-0.2369	0.7914	-0.3432	0.8507	-0.5530	0.9314
10	3	7	0	0.0	1.0000	-0.0500	1.0000	-0.2482	1.0000
10	3	6	3	-1.0000	0.2484	-1.0000	0.3258	-1.0000	0.4973
10	3	6	2	-0.6553	0.4817	-0.7479	0.5662	-0.8806	0.7060
10	3	6	1	-0.3231	0.7487	-0.4228	0.8190	-0.5963	0.9161
10	3	6	0	-0.0466	1.0000	-0.1422	1.0000	-0.3334	1.0000
10	3	5	3	-1.0000	0.1426	-1.0000	0.2324	-1.0000	0.4147
10	3	5	2	-0.7043	0.4031	-0.7854	0.4956	-0.8995	0.6527
10	3	5	1	-0.4031	0.7043	-0.4956	0.7654	-0.6527	0.8995
10	3	5	0	-0.1426	1.0000	-0.2324	1.0000	-0.4147	1.0000
10	4	10	3	-0.3097	1.0000	-0.4676	1.0000	-0.7258	1.0000
10	4	10	2	0.0	1.0000	-0.0472	1.0000	-0.2840	1.0000
10	4	10	1	0.0	1.0000	0.0	1.0000	-0.0109	1.0000
10	4	10	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	4	9	4	-1.0000	0.6518	-1.0000	0.7466	-1.0000	0.8805
10	4	9	3	-0.4811	0.7732	-0.6100	0.8375	-0.8075	0.9251
10	4	9	2	-0.1080	0.8431	-0.2216	0.8887	-0.4322	0.9495
10	4	9	1	0.0	0.9007	0.0	0.9303	-0.1682	0.9687
10	4	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	4	8	4	-1.0000	0.4034	-1.0000	0.5013	-1.0000	0.6617
10	4	8	3	-0.5761	0.5830	-0.6860	0.6555	-0.8485	0.7714
10	4	8	2	-0.2289	0.7086	-0.3344	0.7633	-0.5245	0.8470
10	4	8	1	0.0	0.8559	-0.0917	0.8980	-0.2774	0.9538
10	4	8	0	0.0	1.0000	0.0	1.0000	-0.0430	1.0000
10	4	7	4	-1.0000	0.2761	-1.0000	0.3536	-1.0000	0.5201
10	4	7	3	-0.6426	0.4734	-0.7380	0.5402	-0.8755	0.6679
10	4	7	2	-0.3236	0.6248	-0.4214	0.6902	-0.5935	0.7949
10	4	7	1	-0.0930	0.8170	-0.1899	0.8695	-0.3658	0.9404
10	4	7	0	0.0	1.0000	0.0	1.0000	-0.1403	1.0000
10	4	6	4	-1.0000	0.1469	-1.0000	0.3223	-1.0000	0.4131
10	4	6	3	-0.6948	0.3634	-0.7781	0.4475	-0.8958	0.5894
10	4	6	2	-0.4048	0.5508	-0.4951	0.6253	-0.6505	0.7478
10	4	6	1	-0.1848	0.7790	-0.2781	0.8414	-0.4437	0.9269
10	4	6	0	0.0	1.0000	-0.0960	1.0000	-0.2306	1.0000
10	4	5	4	-1.0000	0.0529	-1.0000	0.1990	-1.0000	0.3197
10	4	5	3	-0.7391	0.2732	-0.8116	0.3623	-0.9124	0.5167
10	4	5	2	-0.4790	0.4790	-0.5617	0.5617	-0.7007	0.7007
10	4	5	1	-0.2732	0.7391	-0.3623	0.8116	-0.5167	0.9124
10	4	5	0	-0.0529	1.0000	-0.1990	1.0000	-0.3197	1.0000

N1	N2	X1	X2	90%		95%		99%	
10	5	10	4	-0.3592	1.0000	-0.5101	1.0000	-0.7511	1.0000
10	5	10	3	0.0	1.0000	-0.1128	1.0000	-0.3424	1.0000
10	5	10	2	0.0	1.0000	0.0	1.0000	-0.0933	1.0000
10	5	10	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	5	10	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	5	9	5	-1.0000	0.6185	-1.0000	0.7208	-1.0000	0.8673
10	5	9	4	-0.5230	0.7427	-0.6439	0.8147	-0.8261	0.9140
10	5	9	3	-0.1741	0.8114	-0.2836	0.8655	-0.4835	0.9385
10	5	9	2	0.0	0.8628	-0.0600	0.9029	-0.2470	0.9561
10	5	9	1	0.0	0.9108	0.0	0.9374	-0.0543	0.9720
10	5	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	5	8	5	-1.0000	0.3512	-1.0000	0.4536	-1.0000	0.6248
10	5	8	4	-0.6125	0.5270	-0.7145	0.6058	-0.8634	0.7347
10	5	8	3	-0.2919	0.6419	-0.3920	0.7048	-0.5700	0.8050
10	5	8	2	-0.0823	0.7409	-0.1772	0.7897	-0.3516	0.8644
10	5	8	1	0.0	0.8702	-0.0043	0.9083	-0.1657	0.9586
10	5	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	5	7	5	-1.0000	0.2042	-1.0000	0.2949	-1.0000	0.4677
10	5	7	4	-0.6744	0.3906	-0.7624	0.4722	-0.8879	0.6091
10	5	7	3	-0.3829	0.5312	-0.4746	0.5999	-0.6340	0.7117
10	5	7	2	-0.1820	0.6644	-0.2724	0.7235	-0.4350	0.8176
10	5	7	1	-0.0461	0.8348	-0.1003	0.8825	-0.2594	0.9465
10	5	7	0	0.0	1.0000	0.0	1.0000	-0.0636	1.0000
10	5	6	5	-1.0000	0.0898	-1.0000	0.2294	-1.0000	0.3503
10	5	6	4	-0.7228	0.2978	-0.7992	0.4093	-0.9063	0.5126
10	5	6	3	-0.4599	0.4400	-0.5437	0.5149	-0.6864	0.6408
10	5	6	2	-0.2709	0.5963	-0.3566	0.6642	-0.5074	0.7751
10	5	6	1	-0.1352	0.8001	-0.1944	0.8570	-0.3447	0.9344
10	5	6	0	0.0	1.0000	0.0	1.0000	-0.1564	1.0000
10	5	5	5	-1.0000	0.0023	-1.0000	0.1007	-1.0000	0.2696
10	5	5	4	-0.7636	0.2162	-0.8298	0.2934	-0.9213	0.4273
10	5	5	3	-0.5296	0.3553	-0.6055	0.4359	-0.7323	0.5746
10	5	5	2	-0.3553	0.5296	-0.4359	0.6055	-0.5746	0.7323
10	5	5	1	-0.2162	0.7636	-0.2934	0.8298	-0.4273	0.9213
10	5	5	0	-0.0023	1.0000	-0.1007	1.0000	-0.2696	1.0000
10	6	10	5	-0.3983	1.0000	-0.5430	1.0000	-0.7703	1.0000
10	6	10	4	-0.0433	1.0000	-0.1638	1.0000	-0.3869	1.0000
10	6	10	3	0.0	1.0000	0.0	1.0000	-0.1536	1.0000
10	6	10	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	6	10	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	6	10	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	6	9	6	-1.0000	0.5896	-1.0000	0.6982	-1.0000	0.8555
10	6	9	5	-0.5555	0.7168	-0.6698	0.7951	-0.8400	0.9044
10	6	9	4	-0.2248	0.7855	-0.3309	0.8463	-0.5218	0.9293

N1	N2	X1	X2	90%		95%		99%	
10	6	9	3	-0.0259	0.8348	-0.1232	0.8825	-0.3035	0.9465
10	6	9	2	0.0	0.8765	0.0	0.9128	-0.1330	0.9606
10	6	9	1	0.0	0.9183	0.0	0.9427	0.0	0.9744
10	6	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	6	8	6	-1.0000	0.3077	-1.0000	0.4136	-1.0000	0.5933
10	6	8	5	-0.6403	0.4817	-0.7362	0.5653	-0.8746	0.7044
10	6	8	4	-0.3397	0.5918	-0.4353	0.6607	-0.6036	0.7730
10	6	8	3	-0.1470	0.6801	-0.2382	0.7368	-0.4039	0.8267
10	6	8	2	-0.0728	0.7640	-0.0772	0.8087	-0.2417	0.8768
10	6	8	1	0.0	0.8809	0.0	0.9159	-0.0867	0.9621
10	6	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	6	7	6	-1.0000	0.1482	-1.0000	0.2469	-1.0000	0.4245
10	6	7	5	-0.6987	0.3350	-0.7809	0.4192	-0.8972	0.5631
10	6	7	4	-0.4272	0.4652	-0.5141	0.5376	-0.6637	0.6585
10	6	7	3	-0.2447	0.5772	-0.3304	0.6396	-0.4832	0.7409
10	6	7	2	-0.1677	0.6931	-0.1755	0.7475	-0.3317	0.8339
10	6	7	1	0.0	0.8482	-0.0332	0.8922	-0.1829	0.9510
10	6	7	0	0.0	1.0000	0.0	1.0000	-0.0152	1.0000
10	6	6	6	-1.0000	0.0445	-1.0000	0.1635	-1.0000	0.2998
10	6	6	5	-0.7440	0.2530	-0.8151	0.3223	-0.9141	0.4549
10	6	6	4	-0.5008	0.3837	-0.5795	0.4648	-0.7126	0.5695
10	6	6	3	-0.3307	0.4917	-0.4111	0.5605	-0.5514	0.6756
10	6	6	2	-0.2595	0.6295	-0.2647	0.6925	-0.4126	0.7948
10	6	6	1	-0.0960	0.8160	-0.1240	0.8687	-0.2724	0.9399
10	6	6	0	0.0	1.0000	0.0	1.0000	-0.1039	1.0000
10	6	5	6	-1.0000	0.0	-1.0000	0.0324	-1.0000	0.1948
10	6	5	5	-0.7820	0.1990	-0.8436	0.2125	-0.9279	0.3611
10	6	5	4	-0.5668	0.3168	-0.6377	0.3548	-0.7553	0.4902
10	6	5	3	-0.4114	0.4114	-0.4862	0.4862	-0.6142	0.6142
10	6	5	2	-0.3168	0.5668	-0.3548	0.6377	-0.4902	0.7553
10	6	5	1	-0.1990	0.7820	-0.2125	0.8436	-0.3611	0.9279
10	6	5	0	0.0	1.0000	-0.0324	1.0000	-0.1948	1.0000
10	7	10	6	-0.4302	1.0000	-0.5696	1.0000	-0.7855	1.0000
10	7	10	5	-0.0864	1.0000	-0.2053	1.0000	-0.4224	1.0000
10	7	10	4	0.0	1.0000	-0.0041	1.0000	-0.2010	1.0000
10	7	10	3	0.0	1.0000	0.0	1.0000	-0.0399	1.0000
10	7	10	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	7	10	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	7	10	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	7	9	7	-1.0000	0.5639	-1.0000	0.6779	-1.0000	0.8449
10	7	9	6	-0.5817	0.6939	-0.6906	0.7777	-0.8510	0.8957
10	7	9	5	-0.2658	0.7633	-0.3687	0.8297	-0.5521	0.9213
10	7	9	4	-0.0773	0.8118	-0.1725	0.8656	-0.3471	0.9385
10	7	9	3	-0.0052	0.8510	-0.0223	0.8943	-0.1906	0.9520
10	7	9	2	0.0	0.8867	0.0	0.9201	-0.0527	0.9640
10	7	9	1	0.0	0.9241	0.0	0.9469	0.0	0.9763
10	7	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000

N1	N2	X1	X2	90%		95%		99%	
10	7	8	7	-1.0000	0.2703	-1.0000	0.3789	-1.0000	0.5657
10	7	8	6	-0.6626	0.4434	-0.7534	0.5308	-0.8833	0.6783
10	7	8	5	-0.3777	0.5509	-0.4696	0.6245	-0.6300	0.7464
10	7	8	4	-0.1971	0.6342	-0.2852	0.6966	-0.4439	0.7979
10	7	8	3	-0.1253	0.7077	-0.1389	0.7598	-0.2965	0.8423
10	7	8	2	0.0	0.7816	-0.0246	0.8231	-0.1629	0.8863
10	7	8	1	0.0	0.8893	0.0	0.9220	-0.0809	0.9649
10	7	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	7	7	7	-1.0000	0.1034	-1.0000	0.2063	-1.0000	0.3875
10	7	7	6	-0.7180	0.2891	-0.7955	0.3753	-0.9044	0.5249
10	7	7	5	-0.4623	0.4133	-0.5452	0.4886	-0.6869	0.6169
10	7	7	4	-0.2926	0.5154	-0.3746	0.5816	-0.5197	0.6919
10	7	7	3	-0.2186	0.6110	-0.2353	0.6688	-0.3832	0.7623
10	7	7	2	-0.0686	0.7152	-0.1242	0.7660	-0.2566	0.8464
10	7	7	1	0.0	0.8587	0.0	0.8999	-0.1608	0.9546
10	7	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	7	6	7	-1.0000	0.0068	-1.0000	0.1115	-1.0000	0.2574
10	7	6	6	-0.7608	0.2236	-0.8277	0.2595	-0.9202	0.4082
10	7	6	5	-0.5328	0.3551	-0.6075	0.3824	-0.7329	0.5158
10	7	6	4	-0.3760	0.4428	-0.4522	0.5053	-0.5845	0.6086
10	7	6	3	-0.3081	0.5300	-0.3219	0.5943	-0.4605	0.7013
10	7	6	2	-0.1635	0.6552	-0.2114	0.7143	-0.3428	0.8099
10	7	6	1	-0.0276	0.8286	-0.0834	0.8778	-0.2292	0.9443
10	7	6	0	0.0	1.0000	0.0	1.0000	-0.0649	1.0000
10	7	5	7	-1.0000	0.0	-1.0000	0.0	-1.0000	0.1492
10	7	5	6	-0.7966	0.1322	-0.8543	0.1655	-0.9331	0.3084
10	7	5	5	-0.5957	0.2639	-0.6626	0.2944	-0.7730	0.4273
10	7	5	4	-0.4535	0.4002	-0.5239	0.4048	-0.6437	0.5340
10	7	5	3	-0.4002	0.4535	-0.4048	0.5239	-0.5340	0.6437
10	7	5	2	-0.2639	0.5957	-0.2944	0.6626	-0.4273	0.7730
10	7	5	1	-0.1322	0.7966	-0.1655	0.8543	-0.3084	0.9331
10	7	5	0	0.0	1.0000	0.0	1.0000	-0.1492	1.0000
10	8	10	7	-0.4570	1.0000	-0.5917	1.0000	-0.7980	1.0000
10	8	10	6	-0.1229	1.0000	-0.2400	1.0000	-0.4517	1.0000
10	8	10	5	0.0	1.0000	-0.0458	1.0000	-0.2397	1.0000
10	8	10	4	0.0	1.0000	0.0	1.0000	-0.0873	1.0000
10	8	10	3	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	8	10	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	8	10	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	8	10	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	8	9	8	-1.0000	0.5407	-1.0000	0.6595	-1.0000	0.8350
10	8	9	7	-0.6034	0.6735	-0.7077	0.7621	-0.8599	0.8879
10	8	9	6	-0.2998	0.7436	-0.4000	0.8150	-0.5768	0.9141
10	8	9	5	-0.1194	0.7920	-0.2127	0.8510	-0.3824	0.9315
10	8	9	4	-0.0638	0.8301	-0.0717	0.8790	-0.2358	0.9448

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N1	N2	X1	X2	90%		95%		99%	
10	8	9	3	0.0	0.8632	0.0	0.9031	-0.1103	0.9561
10	8	9	2	0.0	0.8947	0.0	0.9258	-0.0353	0.9666
10	8	9	1	0.0	0.9289	0.0	0.9502	0.0	0.9778
10	8	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	8	8	8	-1.0000	0.2375	-1.0000	0.3482	-1.0000	0.5409
10	8	8	7	-0.6810	0.4101	-0.7675	0.5006	-0.8904	0.6551
10	8	8	6	-0.4091	0.5161	-0.4978	0.5935	-0.6514	0.7234
10	8	8	5	-0.2377	0.5965	-0.3231	0.6635	-0.4760	0.7738
10	8	8	4	-0.1677	0.6649	-0.1871	0.7226	-0.3390	0.8158
10	8	8	3	-0.0253	0.7289	-0.0937	0.7775	-0.2188	0.8542
10	8	8	2	0.0	0.7957	-0.0136	0.8346	-0.1380	0.8939
10	8	8	1	0.0	0.8961	0.0	0.9268	0.0	0.9671
10	8	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	8	7	8	-1.0000	0.0677	-1.0000	0.1711	-1.0000	0.3552
10	8	7	7	-0.7338	0.2500	-0.8075	0.3377	-0.9103	0.4920
10	8	7	6	-0.4910	0.3702	-0.5705	0.4478	-0.7056	0.5820
10	8	7	5	-0.3311	0.4664	-0.4100	0.5357	-0.5487	0.6534
~~10~~	~~8~~	~~7~~	~~4~~	-0.2595	0.5525	-0.2815	0.6142	-0.4229	0.7166
10	8	7	3	-0.1210	0.6372	-0.1918	0.6915	-0.3099	0.7789
10	8	7	2	0.0	0.7328	-0.1112	0.7808	-0.2377	0.8564
10	8	7	1	0.0	0.8673	0.0	0.9060	-0.1032	0.9575
10	8	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	8	6	8	-1.0000	0.0	-1.0000	0.0686	-1.0000	0.2209
10	8	6	7	-0.7746	0.2094	-0.8380	0.2148	-0.9252	0.3688
10	8	6	6	-0.5589	0.3223	-0.6302	0.3343	-0.7493	0.4721
10	8	6	5	-0.4121	0.4267	-0.4848	0.4339	-0.6106	0.5580
10	8	6	4	-0.3468	0.4872	-0.3656	0.5369	-0.4971	0.6379
10	8	6	3	-0.2145	0.5601	-0.2746	0.6208	-0.3931	0.7215
10	8	6	2	-0.0960	0.6759	-0.1893	0.7318	-0.3115	0.8221
10	8	6	1	0.0	0.8388	-0.0614	0.8853	-0.2192	0.9478
10	8	6	0	0.0	1.0000	0.0	1.0000	-0.0366	1.0000
10	8	5	8	-1.0000	0.0	-1.0000	0.0	-1.0000	0.1104
10	8	5	7	-0.8086	0.0799	-0.8631	0.1276	-0.9373	0.2645
10	8	5	6	-0.6192	0.1990	-0.6827	0.2524	-0.7872	0.3770
10	8	5	5	-0.4868	0.3125	-0.5536	0.3522	-0.6669	0.4739
10	8	5	4	-0.4359	0.4359	-0.4455	0.4455	-0.5671	0.5671
10	8	5	3	-0.3125	0.4868	-0.3522	0.5536	-0.4739	0.6669
10	8	5	2	-0.1990	0.6192	-0.2524	0.6827	-0.3770	0.7872
10	8	5	1	-0.0799	0.8086	-0.1276	0.8631	-0.2645	0.9373
10	8	5	0	0.0	1.0000	0.0	1.0000	-0.1104	1.0000
10	9	10	8	-0.4800	1.0000	-0.6105	1.0000	-0.8084	1.0000
10	9	10	7	-0.1544	1.0000	-0.2699	1.0000	-0.4765	1.0000
10	9	10	6	0.0	1.0000	-0.0814	1.0000	-0.2723	1.0000
10	9	10	5	0.0	1.0000	0.0	1.0000	-0.1267	1.0000
10	9	10	4	0.0	1.0000	0.0	1.0000	-0.0076	1.0000

N1	N2	X1	X2	90%		95%		99%	
10	9	10	3	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	9	10	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	9	10	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	9	10	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	9	9	9	-1.0000	0.5195	-1.0000	0.6425	-1.0000	0.8259
10	9	9	8	-0.6218	0.6549	-0.7221	0.7477	-0.8674	0.8806
10	9	9	7	-0.3289	0.7260	-0.4265	0.8017	-0.5975	0.9075
10	9	9	6	-0.1550	0.7744	-0.2466	0.8379	-0.4118	0.9252
10	9	9	5	-0.1003	0.8120	-0.1125	0.8657	-0.2729	0.9385
10	9	9	4	0.0	0.8438	-0.0238	0.8890	-0.1561	0.9495
10	9	9	3	0.0	0.8728	0.0	0.9100	-0.0809	0.9593
10	9	9	2	0.0	0.9012	0.0	0.9305	0.0	0.9688
10	9	9	1	0.0	0.9328	0.0	0.9530	0.0	0.9791
10	9	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	9	8	9	-1.0000	0.2083	-1.0000	0.3208	-1.0000	0.5185
10	9	8	8	-0.6965	0.3806	-0.7793	0.4737	-0.8964	0.6343
10	9	8	7	-0.4356	0.4856	-0.5215	0.5663	-0.6692	0.7029
10	9	8	6	-0.2716	0.5642	-0.3548	0.6350	-0.5025	0.7529
10	9	8	5	-0.2031	0.6297	-0.2264	0.6918	-0.3736	0.7936
10	9	8	4	-0.0686	0.6886	-0.1498	0.7426	-0.2626	0.8295
10	9	8	3	0.0	0.7458	-0.0626	0.7916	-0.1825	0.8637
10	9	8	2	0.0	0.8073	0.0	0.8441	-0.0577	0.9001
10	9	8	1	0.0	0.9018	0.0	0.9309	0.0	0.9689
10	9	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	9	7	9	-1.0000	0.0364	-1.0000	0.1401	-1.0000	0.3264
10	9	7	8	-0.7472	0.2159	-0.8175	0.3049	-0.9152	0.4629
10	9	7	7	-0.5151	0.3331	-0.5917	0.4127	-0.7211	0.5518
10	9	7	6	-0.3631	0.4255	-0.4393	0.4974	-0.5727	0.6210
10	9	7	5	-0.2934	0.5060	-0.3189	0.5708	-0.4549	0.6805
10	9	7	4	-0.1635	0.5816	-0.2461	0.6397	-0.3514	0.7358
10	9	7	3	-0.0529	0.6585	-0.1594	0.7098	-0.2804	0.7923
10	9	7	2	0.0	0.7475	-0.0490	0.7930	-0.1599	0.8646
10	9	7	1	0.0	0.8744	0.0	0.9112	-0.0457	0.9599
10	9	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	9	6	9	-1.0000	0.0	-1.0000	0.0319	-1.0000	0.1888
10	9	6	8	-0.7861	0.1677	-0.8465	0.1788	-0.9293	0.3345
10	9	6	7	-0.5807	0.2741	-0.6491	0.2943	-0.7629	0.4349
10	9	6	6	-0.4418	0.3689	-0.5117	0.3887	-0.6321	0.5167
10	9	6	5	-0.3787	0.4614	-0.4008	0.4734	-0.5263	0.5899
10	9	6	4	-0.2554	0.5222	-0.3239	0.5625	-0.4318	0.6610
10	9	6	3	-0.1481	0.5847	-0.2579	0.6423	-0.3829	0.7378
10	9	6	2	-0.0433	0.6931	-0.1511	0.7464	-0.2734	0.8321
10	9	6	1	0.0	0.8474	-0.0405	0.8915	-0.1636	0.9507
10	9	6	0	0.0	1.0000	0.0	1.0000	-0.0260	1.0000
10	9	5	9	-1.0000	0.0	-1.0000	0.0	-1.0000	0.0767
10	9	5	8	-0.8186	0.0367	-0.8705	0.0959	-0.9407	0.2269
10	9	5	7	-0.6387	0.1476	-0.6995	0.2205	-0.7989	0.3348

N1	N2	X1	X2	90%		95%		99%	
10	9	5	6	-0.5141	0.2493	-0.5780	0.3151	-0.6859	0.4257
10	9	5	5	-0.4652	0.3510	-0.4780	0.3964	-0.5935	0.5095
10	9	5	4	-0.3510	0.4652	-0.3964	0.4780	-0.5095	0.5935
10	9	5	3	-0.2493	0.5141	-0.3151	0.5780	-0.4257	0.6859
10	9	5	2	-0.1476	0.6387	-0.2205	0.6995	-0.3348	0.7989
10	9	5	1	-0.0367	0.8186	-0.0959	0.8705	-0.2269	0.9407
10	9	5	0	0.0	1.0000	0.0	1.0000	-0.0767	1.0000
10	10	10	9	-0.5000	1.0000	-0.6268	1.0000	-0.8174	1.0000
10	10	10	8	-0.1820	1.0000	-0.2959	1.0000	-0.4979	1.0000
10	10	10	7	-0.0085	1.0000	-0.1123	1.0000	-0.3004	1.0000
10	10	10	6	0.0	1.0000	0.0	1.0000	-0.1602	1.0000
10	10	10	5	0.0	1.0000	0.0	1.0000	-0.0468	1.0000
10	10	10	4	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	10	3	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	10	2	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	10	1	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	10	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	9	10	-1.0000	0.5000	-1.0000	0.6268	-1.0000	0.8173
10	10	9	9	-0.6378	0.6378	-0.7345	0.7345	-0.8738	0.8738
10	10	9	8	-0.3541	0.7099	-0.4494	0.7895	-0.6153	0.9014
10	10	9	7	-0.1857	0.7586	-0.2756	0.8261	-0.4369	0.9194
10	10	9	6	-0.1317	0.7960	-0.1471	0.8538	-0.3042	0.9328
10	10	9	5	0.0	0.8271	-0.0691	0.8767	-0.1940	0.9437
10	10	9	4	0.0	0.8546	0.0	0.8968	-0.1187	0.9531
10	10	9	3	0.0	0.8805	0.0	0.9155	-0.0021	0.9619
10	10	9	2	0.0	0.9066	0.0	0.9343	0.0	0.9705
10	10	9	1	0.0	0.9362	0.0	0.9554	0.0	0.9801
10	10	9	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	8	10	-1.0000	0.1820	-1.0000	0.2959	-1.0000	0.4979
10	10	8	9	-0.7099	0.3541	-0.7895	0.4494	-0.9014	0.6153
10	10	8	8	-0.4585	0.4585	-0.5418	0.5418	-0.6844	0.6844
10	10	8	7	-0.3007	0.5358	-0.3818	0.6098	-0.5250	0.7343
10	10	8	6	-0.2335	0.5994	-0.2596	0.6652	-0.4025	0.7744
10	10	8	5	-0.1050	0.6554	-0.1982	0.7137	-0.2985	0.8089
10	10	8	4	0.0	0.7077	-0.1028	0.7587	-0.2192	0.8405
10	10	8	3	0.0	0.7598	0.0	0.8033	-0.1124	0.8716
10	10	8	2	0.0	0.8170	0.0	0.8521	-0.0174	0.9053
10	10	8	1	0.0	0.9066	0.0	0.9343	0.0	0.9705
10	10	8	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	7	10	-1.0000	0.0085	-1.0000	0.1123	-1.0000	0.3004
10	10	7	9	-0.7586	0.1857	-0.8261	0.2756	-0.9194	0.4369
10	10	7	8	-0.5358	0.3007	-0.6098	0.3818	-0.7343	0.5250
10	10	7	7	-0.3903	0.3903	-0.4642	0.4642	-0.5928	0.5928
10	10	7	6	-0.3223	0.4670	-0.3502	0.5344	-0.4815	0.6501
10	10	7	5	-0.1990	0.5371	-0.2926	0.5984	-0.3851	0.7017
10	10	7	4	-0.0960	0.6052	-0.1986	0.6604	-0.3152	0.7515
10	10	7	3	-0.0015	0.6762	-0.0978	0.7251	-0.2066	0.8035

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N1	N2	X1	X2	90%		95%		99%	
10	10	7	2	0.0	0.7598	0.0	0.8033	-0.1124	0.8716
10	10	7	1	0.0	0.8805	0.0	0.9155	-0.0021	0.9619
10	10	7	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	6	10	-1.0000	0.0	-1.0000	0.0	-1.0000	0.1602
10	10	6	9	-0.7960	0.1317	-0.8538	0.1471	-0.9328	0.3042
10	10	6	8	-0.5994	0.2335	-0.6652	0.2596	-0.7744	0.4025
10	10	6	7	-0.4670	0.3223	-0.5344	0.3502	-0.6501	0.4815
10	10	6	6	-0.4057	0.4057	-0.4299	0.4299	-0.5505	0.5505
10	10	6	5	-0.2894	0.4899	-0.3643	0.5048	-0.4631	0.6152
10	10	6	4	-0.1902	0.5510	-0.2954	0.5839	-0.4183	0.6799
10	10	6	3	-0.0960	0.6052	-0.1986	0.6604	-0.3152	0.7515
10	10	6	2	0.0	0.7077	-0.1028	0.7587	-0.2192	0.8405
10	10	6	1	0.0	0.8546	0.0	0.8968	-0.1187	0.9531
10	10	6	0	0.0	1.0000	0.0	1.0000	0.0	1.0000
10	10	5	10	-1.0000	0.0	-1.0000	0.0	-1.0000	0.0468
10	10	5	9	-0.8271	0.0	-0.8767	0.0691	-0.9437	0.1940
10	10	5	8	-0.6554	0.1050	-0.7137	0.1982	-0.8089	0.2985
10	10	5	7	-0.5371	0.1990	-0.5984	0.2926	-0.7017	0.3851
10	10	5	6	-0.4899	0.2894	-0.5048	0.3643	-0.6152	0.4631
10	10	5	5	-0.3828	0.3828	-0.4318	0.4318	-0.5380	0.5380
10	10	5	4	-0.2894	0.4899	-0.3643	0.5048	-0.4631	0.6152
10	10	5	3	-0.1990	0.5371	-0.2926	0.5984	-0.3851	0.7017
10	10	5	2	-0.1050	0.6554	-0.1982	0.7137	-0.2985	0.8089
10	10	5	1	0.0	0.8271	-0.0691	0.8767	-0.1940	0.9437
10	10	5	0	0.0	1.0000	0.0	1.0000	-0.0468	1.0000

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Odds ratio, relative risk, difference of success probabilities, contingency tables, frequential approach

P sub 1 P sub 2 Pi sub 1 Pi sub 2

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Consider two binomial populations Π_1 and Π_2 having success probabilities p_1 in (0,1) and p_2 in (0,1) respectively. This paper studies the problem of constructing exact small sample confidence intervals for the difference of the success probabilities, $\Delta \equiv p_1-p_2$ and their ratio → next page

delta is nearly = to P sub 1 to P sub 2

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Rho is nearly = to P sub 1 / P sub 2

cont.

(the "relative risk"), $\rho \equiv p_1/p_2$ based on independent random samples of sizes N_1 (N sub 1) and N_2 (N sub 2) from Π_1 (Pi sub 1) and Π_2 (Pi sub 2) respectively. These are nuisance parameter problems; hence the proposed intervals achieve coverage probabilities greater than or equal to their nominal $(1-\alpha)$ (alpha) levels. Two methods of constructing intervals are proposed. The first one is based on the well known conditional intervals for the odds ratio. $\psi \equiv p_1(1-p_2)/p_2(1-p_1)$. It yields easily computable Δ and ρ intervals. Tables are provided. The second method directly generates unconditional intervals of the desired size. An algorithm is given for producing the intervals for arbitrary N_1 (N sub 1) and N_2 (N sub 2). The 2×2 case is given as an illustrative example. Some comparisons are made.